TECHNISCHE UNIVERSITÄT DRESDEN

A GENTLE INTRODUCTION TO CLONES, THEIR GALOIS THEORY, AND APPLICATIONS

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Bath, UK, 1 March 2013

Outline

Vocabulary

- an algebra $\mathbf{A} = \langle A; F \rangle$ = a set with finitary operations ($\approx$ an algebra for a certain monad)
- an algebra may have a signature or not
- signature = a set of operation symbols + a function assigning arities to them
- variety = equational theory, a class of algebras of the same signature, fulfilling a given set of identities
- n-ary term = an element of the free algebra with n generators in the category of all algebras of a given signature
- n-ary term op of $\mathbf{A}$ = an element of the free algebra with n generators in the variety generated by $\mathbf{A}$.

Outline

Assumptions

- Everything in $\mathcal{S}et$!!!
- functions: $A, B \in \mathcal{S}et \quad f\colon A \longrightarrow B$
- $B^A := \mathcal{S}et(A, B)$
- natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$
- $n = \{0, \ldots, n-1\}$ for $n \in \mathbb{N}$

Finitary operations

Let A be a set, $n \in \mathbb{N}$ an integer, called arity.

- $$\begin{aligned} f\colon \quad A^n &\longrightarrow A \\ (x_0, \ldots, x_{n-1}) &\longmapsto f(x_0, \ldots, x_{n-1}) \end{aligned}$$
  n-ary function (operation) on A
- $\mathrm{O}_A^{(n)} := A^{A^n}$ set of all n-ary operations
- set of all finitary operations
  $\mathrm{O}_A := \bigcup_{k\in\mathbb{N}} \mathrm{O}_A^{(k)} = \left\{ f\colon A^k \longrightarrow A \;\middle|\; k \in \mathbb{N} \right\}$

Special (trivial) operations

- projections for $n \in \mathbb{N}_+$, and $0 \leq i < n$

$$e_i^{(n)} : \begin{array}{ccc} A^n & \longrightarrow & A \\ (x_0, \ldots, x_{n-1}) & \longmapsto & x_i \end{array}$$

- set of all projections $\mathrm{J}_A := \left\{ e_i^{(k)} \;\middle|\; k \in \mathbb{N}, 0 \leq i < k \right\}$

Composition of operations

For $n, k \in \mathbb{N}$ $f \in \mathrm{O}_A^{(n)}$ and $(g_0, \ldots, g_{n-1}) \in \left(\mathrm{O}_A^{(k)}\right)^n$:

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Input: $(a_0, \ldots, a_{n-1}) \in A^n$

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For $n, k \in \mathbb{N}$ $f \in \mathrm{O}_A^{(n)}$ and $(g_0, \ldots, g_{n-1}) \in \left(\mathrm{O}_A^{(k)}\right)^n$:

Output: $f(a_0, \ldots, a_{n-1}) \in A$

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$$\text{Output:} \qquad f(\overset{g_0}{\overset{\downarrow}{a_0}}, \ldots, \overset{g_{n-1}}{\overset{\downarrow}{a_{n-1}}}) \in A$$

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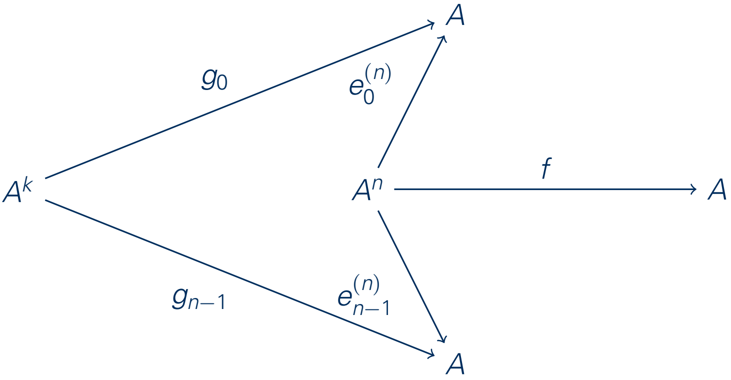
Composition of operations

For $n, k \in \mathbb{N}$ $f \in \mathrm{O}_A^{(n)}$ and $(g_0, \ldots, g_{n-1}) \in \left(\mathrm{O}_A^{(k)}\right)^n$:

$$\begin{array}{rccc} f \circ \langle g_0, \ldots, g_{n-1} \rangle : & A^k & \longrightarrow & A \\ & \mathbf{a} = (a_0, \ldots, a_{k-1}) & \longmapsto & f(g_0(\mathbf{a}), \ldots, g_{n-1}(\mathbf{a})) \end{array}$$

Composition illustrated



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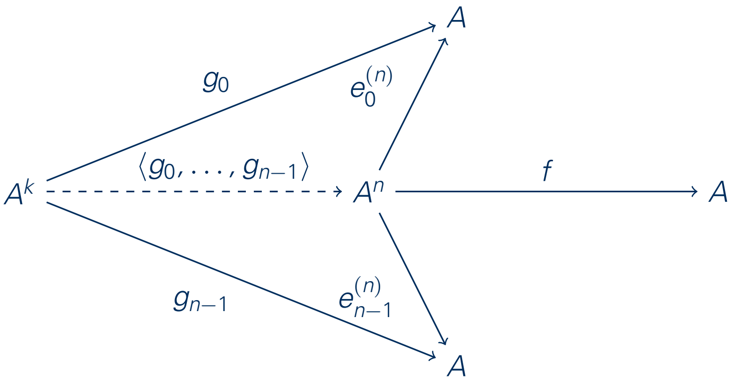

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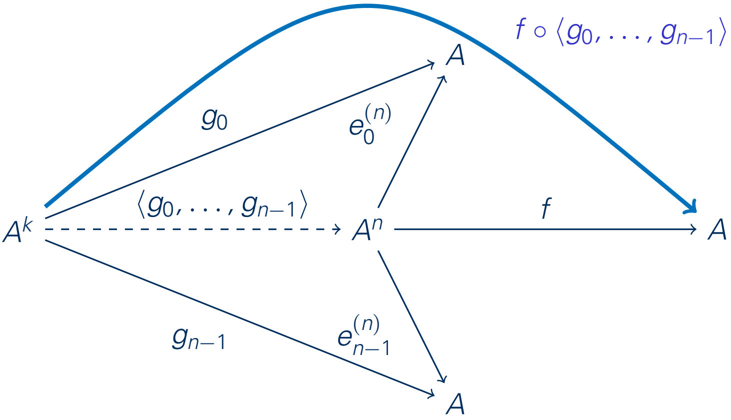

Clones

Definition (   )

$F \subseteq O_A$ (conrete) clone (of operations) on A iff

1. $J_A \subseteq F$
2. F is closed w.r.t. composition

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- J_A clone of all projections
- O_A clone of all operations

Examples of clones (I)

- $\mathrm{O}^{c}_{(A,\tau)}$ continuous functions of a topological space (A,τ)
- $\mathrm{O}^{m}_{(A,\Sigma)}$ measurable functions of a measurable space (A,Σ)
- $\mathrm{O}^{h}_{(A,F)}$ all homomorphisms from direct powers of an algebra (A,F) to itself.
- Pick an object A in a familiar concrete category $\mathcal{C}$ having finite powers and forgetful functor $U\colon \mathcal{C} \longrightarrow \mathcal{S}et$. Consider $\{\, U(f) \mid f \in \mathcal{C}(A^n, A)\,, n \in \mathbb{N}\}$

Prototypical construction (I)

Lemma

$(\mathcal{C}, U \colon \mathcal{C} \longrightarrow \mathcal{S}et)$ concrete category over $\mathcal{S}et$, having finite products preserved by U, $A \in \mathcal{C}$, then

$$\mathrm{O}_A^{\mathcal{C}} := \bigcup_{n\in\mathbb{N}} \{\, U(f) \mid f \in \mathcal{C}\,(A^n, A)\}$$

is a clone, the clone over A.
Every clone can be given this way.

Examples of clones (II)

- $F \subseteq \mathrm{O}_A \rightsquigarrow$ close $F \cup \mathrm{J}_A$ w.r.t. composition (clones form closure system).
- $\mathbf{A} = \langle A; F\rangle \rightsquigarrow$ compute all term operations of $\mathbf{A}$.
- $A = \{0,1\}$, $F = \{\min\}$,
  $\rightsquigarrow \mathrm{Clo}(\mathbf{A}) =$
  $$\left\{ (a_0,\dots,a_{n-1}) \mapsto \min\{a_i \mid i \in I\} \;\middle|\; \begin{array}{l} \emptyset \neq I \subseteq \{0,\dots,n-1\}, \\ n \in \mathbb{N}_+ \end{array} \right\}$$
- $A = \{0,1\}$ $F = \{f\}$, $f(x) = 1 + x \bmod 2$.
  $\mathrm{Clo}(\mathbf{A}) = \mathrm{J}_A \cup \{(a_0,\dots,a_{n-1}) \mapsto f(a_i) \mid 0 \leq i < n, n \in \mathbb{N}_+\}$
- $A = \{0,1\}$ $F = \left\{f, \min, c_0^{(0)}\right\}$, then $\mathrm{Clo}(\mathbf{A}) = \mathrm{O}_A$

Prototypical construction (II)

$\mathbf{A} = \langle A; F \rangle$, where $F \subseteq \mathrm{O}_A$; $f \in F$ function $\leftrightarrow$ symbol
$\rightsquigarrow$ canonical signature $\rightsquigarrow$ $\mathcal{V}$ equational theory generated by $\mathbf{A}$
$\rightsquigarrow$ $\mathrm{Clo}\,(\mathbf{A})$ all term op's in this signature $(= \bigcup_{n \in \mathbb{N}} \mathcal{F}_{\mathcal{V}}(n))$

Proposition

For every algebra $\mathbf{A} = \langle A; F \rangle$ *its set of term operations* $\mathrm{Clo}\,(\mathbf{A})$ *forms a clone.*
Every clone $F \subseteq \mathrm{O}_A$ *is the set of term operations of some algebra, namely that of* $\mathbf{A} = \langle A; F \rangle$*.*

Importance of clones (for universal algebra)

- Goal of universal algebra: study structures $\mathbf{A} = \langle A; F \rangle$, where $F \subseteq \mathrm{O}_A$. (difficult)
- Important properties of algebras only depend on $\mathrm{Clo}\,(\mathbf{A})$, not on F (e.g. substructures, homomorphisms, congruences, ...).
- In principle, study of $\langle A; F \rangle$ such that F clone suffices

Examples of clones (III)

Let A be a set, $a \in A$ and $U \subseteq A$.

- $T_a := \{f \in \mathrm{O}_A \mid f(a, \ldots, a) = a\}$ a-preserving operations
- $F_U := \left\{f \in \mathrm{O}_A \;\middle|\; f\left[U^{\operatorname{ar} f}\right] \subseteq U\right\}$ U-preserving operations
- $\bigcap_{a\in A} T_a$ clone of all idempotent op's ($f(x, \ldots, x) \approx x$)
- $\bigcap_{U\subseteq A} F_U$ clone of all conservative op's ($f(x_0,\ldots,x_{n-1}) \in \{x_0,\ldots,x_{n-1}\}$)
- All monotone operations w.r.t. an order $\leq$ on A:

$$\begin{matrix} a_0 & \cdots & a_{n-1} \\ \leq & \cdots & \leq \\ b_0 & \cdots & b_{n-1} \end{matrix} \quad\Longrightarrow\quad \begin{matrix} f(a_0, \ldots, a_{n-1}) \\ \leq \\ f(b_0, \ldots, b_{n-1}). \end{matrix}$$

- All operations preserving every member of a given set of finitary relations on A.

Prototypical construction (III)

Lemma

Every set of operations described by preserving each relation of a given set of finitary relations on A forms a clone on A.

Question

Which clones arise in this way?

Outline

Finitary relations

- For $m \in \mathbb{N}$ subsets $\varrho \subseteq A^m$ are m-ary relations on A
- Elements $(x_0, \ldots, x_{m-1}) \in \varrho$ also interpretable as $x\colon m \longrightarrow A$
- $\mathrm{R}_A^{(m)} := \mathcal{P}(A^m)$ set of m-ary relations on A
- $\mathrm{R}_A := \bigcup_{m \in \mathbb{N}} \mathrm{R}_A^{(m)}$ set of all finitary relations on A

Preservation (compatibility) relation

$m, n \in \mathbb{N}$, $f \in \mathrm{O}_A^{(n)}$, $\varrho \in \mathrm{R}_A^{(m)}$

$$\begin{matrix} x_{0,0} & \cdots & x_{0,n-1} \\ \vdots & \ddots & \vdots \\ x_{m-1,0} & \cdots & x_{m-1,n-1} \end{matrix}$$

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 & \vdots & \ddots & \vdots & \\
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\end{array}$$

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$$

Truth of this condition: $f \rhd \varrho$

One way to understand preservation

Let $m, n \in \mathbb{N}$, $f \in \mathrm{O}_A^{(n)}$, $\varrho \in \mathrm{R}_A^{(m)}$.

$$f \rhd \varrho \iff \forall\, r_0, \dots, r_{n-1} \in \varrho \colon f \circ \langle r_0, \dots, r_{n-1} \rangle \in \varrho$$

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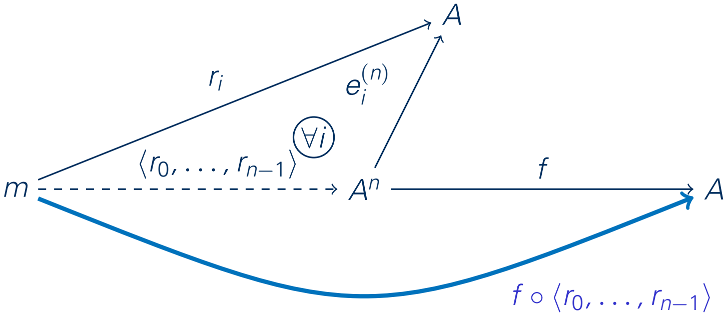

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$$\varrho \overset{\iota}{\rightarrowtail} A^m$$

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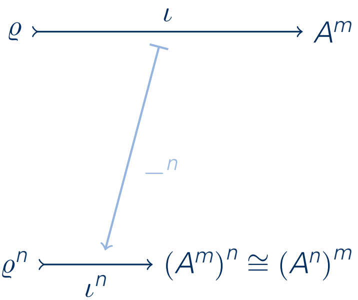

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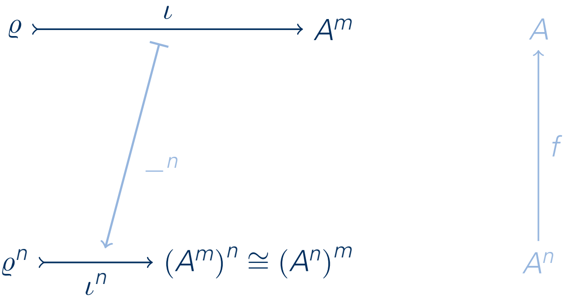

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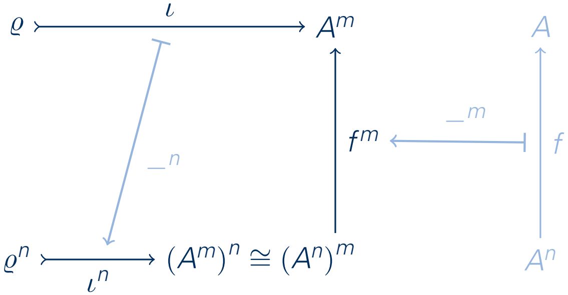

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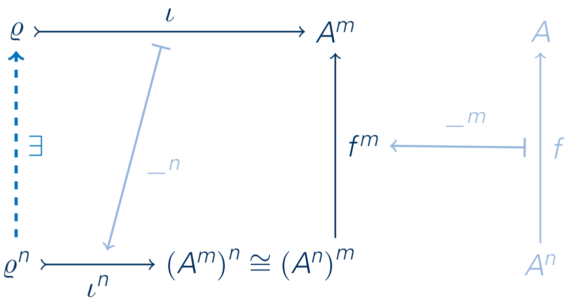

Polymorphisms and invariant relations

For $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$:

$$\mathrm{Inv}\,\langle A; F\rangle := \mathrm{Inv}_A\, F := \{\,\varrho \in \mathrm{R}_A \mid \forall f \in F: \quad f \rhd \varrho\}$$
$$\mathrm{Pol}\,\langle A; Q\rangle := \mathrm{Pol}_A\, Q := \{\,f \in \mathrm{O}_A \mid \forall \varrho \in Q: \quad f \rhd \varrho\}$$

closure operators

$$F \mapsto \mathrm{Pol}_A\,\mathrm{Inv}_A\, F$$
$$Q \mapsto \mathrm{Inv}_A\,\mathrm{Pol}_A\, Q$$

What are the closures?

What is

$$\{ F \subseteq \mathrm{O}_A \mid F = \mathrm{Pol}_A \,\mathrm{Inv}_A\, F\} = \{ \mathrm{Pol}_A\, Q \mid Q \subseteq \mathrm{R}_A\}$$

$$\{ Q \subseteq \mathrm{R}_A \mid Q = \mathrm{Inv}_A \,\mathrm{Pol}_A\, Q\} = \{ \mathrm{Inv}_A\, F \mid F \subseteq \mathrm{O}_A\}?$$

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Lemma

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Consequence for $F \subseteq \mathrm{O}_A$

$F \subseteq \mathrm{Pol}_A \mathrm{Inv}_A F$ clone $\implies \mathrm{Clo}(\langle A; F\rangle) \subseteq \mathrm{Pol}_A \mathrm{Inv}_A F$

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Which clones arise this way?

Local closure

Given $g \in O_A^{(n)}$, and $F \subseteq O_A^{(n)} = A^{(A^n)}$.

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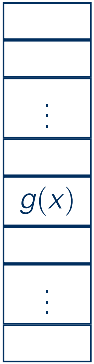

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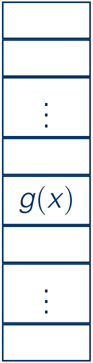

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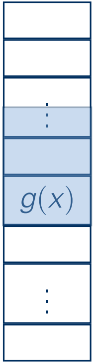

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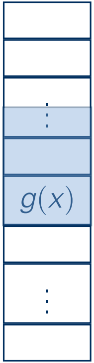

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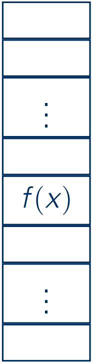

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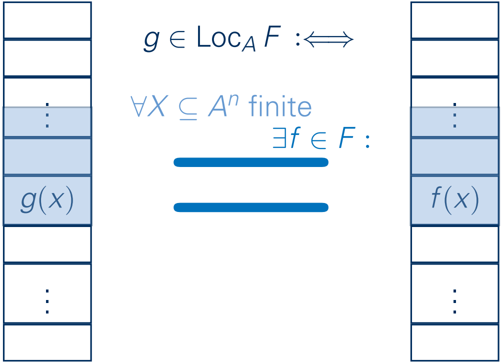

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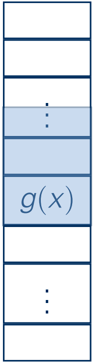

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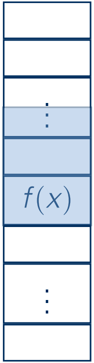
$\exists f \in F :$

$f|_X = g|_X$



Local closure is the answer

Theorem (Corollary (Pöschel 79/80))

For $F \subseteq \mathrm{O}_A$ it is $\mathrm{Pol}_A \mathrm{Inv}_A F = \mathrm{Loc}_A \mathrm{Clo}(\langle A; F \rangle)$.

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Corollary (Bodnarčuk, Kalužnin, Kotov, Romov 69, Geiger 68)

A finite $\implies \{\mathrm{Pol}_A Q \mid Q \subseteq \mathrm{R}_A\} =$ set of all clones.

The other side?

What is

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Local closures of relational clones

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Relational clones?

They have got an intrinsical description, but technical!

Outline

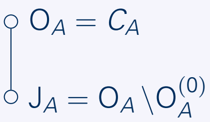
Structure of all clones

- $\mathcal{L}_A$ set of all clones on A
- $(\mathcal{L}_A, \subseteq)$ poset, even complete algebraic lattice.
- $\mathcal{L}_A$ closed under $\bigcap$, i.e. closure system
- closure operator $\mathrm{O}_A \supseteq F \mapsto \langle F \rangle$ =least clone containing F = closure of $F \cup \mathrm{J}_A$ against composition = $\mathrm{Clo}\left(\langle A; F\rangle\right)$.
- $\bigvee \mathcal{F} = \langle \bigcup \mathcal{F} \rangle$

Clone lattice on $A = \emptyset$

- $\mathrm{O}_A = \mathrm{J}_A = \mathcal{C}_A$

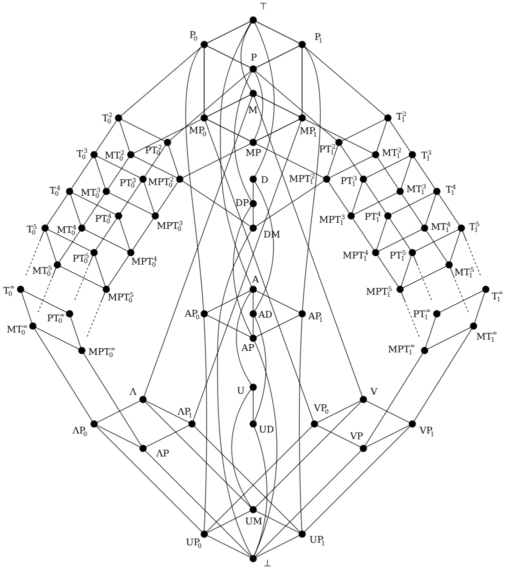
Clone lattice on $A = \{0\}$



Clone lattice on $A = \{0, 1\}$ (without $O_A^{(0)}$)

Emil Leon Post (1897 – 1954)

Clone lattice on $A = \{0, 1\}$ (without $O_A^{(0)}$)



Clone lattice for $3 \leq |A| < \aleph_0$

Draw a picture!

Clone lattice for $3 \leq |A| < \aleph_0$

Theorem (Janov, Muchnik, 1959)

There exist $\aleph_1$-many clones on any set A having at least three distinct elements.

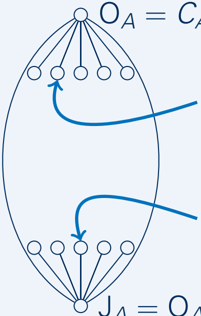
Corollary

$|\mathcal{L}_A| = \aleph_1$ *for* $3 \leq |A| < \aleph_0$.

Minimal and maximal clones

Usually for finite A.

$\mathrm{O}_A = C_A$



coatoms = "maximal clones"

$F = \mathrm{Pol}_A\{\varrho\}$, ϱ Rosenberg relation

atoms = "minimal clones"

$F = \mathrm{Clo}(\langle A; f\rangle)$, f minimal function

$\mathrm{J}_A = \mathrm{O}_A \setminus \mathrm{O}_A^{(0)}$

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Classification of minimal functions

Theorem (I.G. Rosenberg, 1983, 1986)

1. *Every minimal function $f \in \mathrm{O}_A$ ($2 \leq |A| < \aleph_0$) satisfies one of the following conditions*
   1. $f \in \mathrm{O}_A^{(1)}$ *and* ($f^2 = f$ *or* $f^p = \mathrm{id}_A$ *for a prime* p).
   2. $f \in \mathrm{O}_A^{(2)}$ *idempotent, i.e.* $\forall\, x \in A\colon f(x,x) = x$.
   3. $f \in \mathrm{O}_A^{(3)}$ *majority operation, i.e.* $\forall\, x,y \in A\colon\; f(x,x,y) = f(x,y,x) = f(y,x,x) = x$.
   4. $f \in \mathrm{O}_A^{(3)}$ *and* $\forall\, x,y,z \in A\colon f(x,y,z) = x+y+z$, *where* $+ \in \mathrm{O}_A^{(2)}$ *is a* BOOLE*an group operation, i.e.* $\forall\, x \in A\colon x + x = 0$.
   5. $f \in \mathrm{O}_A^{(n)}$ $3 \leq n$*-ary semiprojection, i.e.*

$$\exists\, 1 \leq i \leq n\, \forall\, x \in A^n\colon |\operatorname{im} x| < n \implies f(x) = x(i).$$

2. *Cases (1) and (4): conditions even sufficient for minimality of* $\mathrm{Clo}\,(\langle A;\ f\rangle)$.

Maximal clones

Theorem (I.G. Rosenberg, 1970)

A finite, $F \leq \mathrm{O}_A$ *maximal iff* $F = \mathrm{Pol}_A\{\varrho\}$*, for* ϱ

1. *partial order with least and greatest element.*
2. *graph* $\{(x, f(x)) \mid x \in A\}$ *of prime permutation* f
3. *non-trivial equivalence relation on* A*.*
4. *affine relation w.r.t. some elementary* ABEL*ian* p*-group on* A*,* p *prime.*
5. *a central relation of arity* h *(*$1 \leq h < |A|$*).*
6. *an* h*-regular relation (*$3 \leq h \leq |A|$*).*

Outline

CSP (Constraint Satisfaction Problem)

- $\Sigma =$ a finite relational language,
- $\mathcal{R}el\,(\Sigma)$ category of all relational structures in language Σ
- $\mathcal{R}el_{fin}\,(\Sigma)$ full subcategory of $\mathcal{R}el\,(\Sigma)$: objects with finite carrier sets

Definition

For $\mathbb{B} \in \mathcal{R}el_{fin}\,(\Sigma)$, $\mathrm{CSP}(\mathbb{B})$ is the membership (decision) problem for the following set: $\left\{ \mathbb{A} \in \mathcal{R}el_{fin}\,(\Sigma) \;\middle|\; \mathbb{A} \longrightarrow \mathbb{B} \right\}$

Facts

Lemma

For $\mathbb{B} \in \mathcal{Rel}_{fin}(\Sigma)$, $\mathrm{CSP}(\mathbb{B})$ *is in* NP.

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For which $\mathbb{B} \in \mathcal{Rel}_{fin}(\Sigma)$ is $\mathrm{CSP}(\mathbb{B})$ in P?

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CSP-Dichotomy Conjecture (Feder,Vardi, 1998)

For every $\mathbb{B} \in \mathcal{R}el_{fin}(\Sigma)$, $\mathrm{CSP}(\mathbb{B})$ either is in P or it is NP-complete.

Where does clone theory come into play?

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Proposition

If $\mathbb{B}, \mathbb{C} \in \mathcal{Rel}_{fin}(\Sigma)$ *have the same carrier and* Pol $\mathbb{B} \subseteq$ Pol $\mathbb{C}$, *then* $\mathrm{CSP}(\mathbb{C})$ *is polynomial-time reducible to* $\mathrm{CSP}(\mathbb{B})$, *i.e.,* $\mathrm{CSP}(\mathbb{C})$ *is at most as hard as* $\mathrm{CSP}(\mathbb{B})$.

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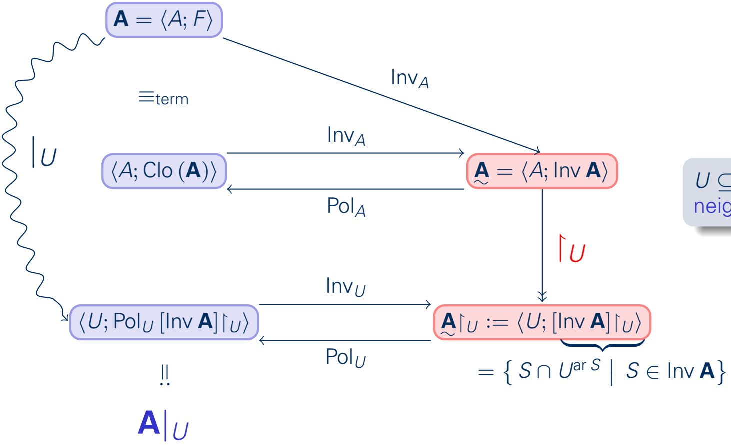
Recent research

One proves that $\mathrm{CSP}(\mathbb{B})$ belongs to P by finding nice polymorphisms in Pol $\mathbb{B}$, e.g. semilattice, nu, Mal'cev, totally symmetric idempotent, edge operations . . .

A relational decomposition theory for algebras (RST)



$U \subseteq A$ special subset, neighbourhood.