Foundations of mathematics

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1. First foundations of mathematics

1.1. Introduction to the foundations of mathematics

Mathematics, theories and foundations

Mathematics is the study of systems of elementary objects, whose only nature is to be exact, unambiguous (two objects are equal or different, related or not; an operation gives an exact result. . .). Such systems are conceived independently of our usual world, even if many of them can resemble (thus be used to describe) diverse aspects of the world. Mathematics as a whole can be seen as "the science of all possible worlds" of its own kind (of exact objects).

It is split into diverse branches, implicit or explicit frameworks of any mathematical work, that may be formalized as (axiomatic) theories. Each theory is the study of a supposedly fixed system (world of objects), called its *model*. But each model of a theory may be just one of its possible interpretations, among other equally legitimate models. For example, roughly speaking, all sheets of paper are systems of material points, models of the same theory of Euclidean plane geometry, but independent of each other. Each theory starts with a *foundation*, that is the data of a (list of pieces of) description specifying what it knows or assumes of its model (its kind or shape). This includes a list of formulas (claims) called *axioms*, selecting its accepted models as the systems where the axioms are true, from the whole range of possible systems where they can be interpreted.

The study of a theory progresses by choosing some of its possible developments : new concepts and information about its models, resulting from its given foundation, and that we can add to it to form its next foundation. In particular, a *theorem* of a theory, is a formula deduced from the axioms, so that it is known as true for all its models, and can be added to the list of axioms without modifying the meaning of the theory. Other possible developments not yet chosen can still come later, as the pieces of foundation that could generate them are preserved. Thus, the totality of possible developments of a theory already forms a kind of "reality" that these developments explore (until the Completeness theorem in 3.1 will show how the range of possible theorems precisely fit the more interesting reality of the range of possible models).

A fundamental work is to develop, from a simple initial basis, a more complete foundation providing efficient tools for further interesting developments. There are possible hierarchies between theories, where some can play a foundational role for others. For instance, several theories may have a common part forming a simpler theory, whose developments are applicable to all.

The cycle of foundations

Despite the simplicity of mathematical objects, the general foundation of all mathematics turns out to be quite complex (though not as bad as a physics'theory of everything). Indeed, it is itself a mathematical study, thus a branch of mathematics, called *mathematical logic*. Like any other branch, it is made of definitions and theorems about systems of objects (the general form of theories and systems to be studied). But moreover, it provides the frameworks of all branches of mathematics. . . including itself. So, to provide the foundation (preliminaries, framework) of each considered foundation (unlike ordinary mathematical works that go forward from an assumed foundation), it does not form a starting point, but a sort of wide cycle composed of easier and harder steps.

(This is similar to dictionaries defining each word by other words, or to another science of finite systems: computer programming. Indeed computers can be simply used, knowing what you do but not why it works; their working is based on software that was written in some language, then compiled by other software, and on the hardware and processor whose design and production were computer assisted. And this is much better than at the birth of this field.)

Still this cycle is a true foundation, providing rigorous frameworks and many useful concepts (tools, inspirations and answers to philosophical questions) to the diverse branches of mathematics. It is dominated by two theories:

Set theory studies the universe of "all mathematical objects", from the simplest to the most complex such as infinite systems. It can roughly be seen as one theory, but turns out to have an illimited diversity of possible variants (not always equivalent to each other) when studied in details.

Model theory is the general theory of theories (describing them as systems of symbols), and their models (systems of objects they may describe).

Each one is the natural framework to formalize the other: each set theory is formalized as a theory described by model theory; model theory better comes as a development from set theory (defining theories and systems as complex objects) than directly as a theory. Both connections must be done separately: both roles of set theory, as a framework and an object of study for model theory, must be distinguished. But these formalizations will be hard to complete, especially for this following last piece:

Proof theory completes model theory by describing the rules of proofs giving the theorems of any theory. A theory is *consistent* if its theorems will never contradict each other. Inconsistent theories cannot have any model, as the same claim cannot be true and false on the same system. Model theory and proof theory are essentially unique, giving a clear meaning to the concepts of theory, theorems and consistency of each theory.

1.2. Variables, sets, functions and operations

To start mathematics, we need to introduce some simple concepts (in the founding cycle), which may seem self-sufficient. It is natural to start with a set theory that is not fully formalized as an axiomatic theory. Let us first explain what is a set, then we will complete the picture with more concepts and explanations on the context of foundations (model theory) and its main subtleties (paradoxes).

Constants

A *constant symbol* is a symbol denoting a unique object, called its *value*. Examples: 3, $\emptyset$, $\mathbb{N}$. Those of English language are proper names and names with "the" (in singular without complement).

Free and bound variables

A *variable symbol* (or a *variable*), is a symbol without a fixed interpretation. Each possible interpretation gives it a particular value and thus sees it as a constant.

It can be understood as limited by a box, whose inside has multiple versions in parallel. From inside, the variable is seen as having a fixed value, thus usable as a constant : it is called *fixed*. We call the variable *free* when we start going out and find that this fixed value is not uniquely determined as the only possible one. These diverse internal "viewpoints", corresponding to the possible values, may be thought of as abstract "locations" in the mathematical universe.

A variable is called *bound* when completely seen from its outside, where the diversity of its possible values is considered as exhaustively known (perceived), the information from them is fully gathered and ready to be processed as a whole.

This succession of viewpoints to a symbol (statuses as constant, free variable, or bound variable), can be seen as a first expression of the flow of time in mathematics: a variable is bound when all the diverse internal locations are past. All these places and times are themselves purely abstract, mathematical entities.

The variability of the model

For example, each consistent theory assumes its model as fixed, but this is usually a mere choice of one model in a wide (infinite) range of other "existing", equally legitimate models of the same theory; the model becomes variable when viewed by model theory. But, this "choice" and this "existence" of a model can be quite abstract. In details, the proof of the Completeness theorem, in the way it can work in all cases, will effectively "specify" a model in the range of possibilities, but this construction is not really explicit as it will involve an infinity of steps, where each step depends on an infinite knowledge. In these conditions, the assumption of fixation of a model may be called nonsense, but nevertheless constitutes the standard interpretation of mathematical theories.

Ranges and sets

The *range* of a variable, is its new meaning when seen as bound: it is the "knowledge" of the totality of its possible or authorized values (seen in bulk: unordered, ignoring their context), that are called the *elements* of this range. This "knowledge" is an abstract entity that can encompass infinities of objects, unlike human thought. A variable has a range if it can be bound, i.e. when an encompassing view over all its possible values is given. Any range of a variable is called a *set*.

A variable is said to *range over* a set, when it is bound with this set as its range. Any number of variables can be introduced with a given range, independently of each other and of other variables.

Cantor defined a set as "*a grouping into a whole of distinct objects of our intuition or our thought*". He explained to Dedekind : "*If the totality of elements of a multiplicity can be thought of as "simultaneously existing", so that it can be conceived as a "single object" (or "completed object"), I call it a consistent multiplicity or a "set".*" (We expressed this "multiplicity" as that of values of a variable).

He described the opposite case as an "*inconsistent multiplicity*" where "*admitting a coexistence of all its elements leads to a contradiction*". But non-contradiction cannot suffice to generally define sets: the non-contradiction of a claim does not imply its truth (the opposite claim may be true but unprovable); facts of non-contradiction can be themselves unprovable (Incompleteness theorem); and two separately consistent coexistences might contradict each other (Irresistible force paradox(*)).

Systematically renaming a bound variable in all its box, into another symbol not used in the same context (same box), with the same range, does not change the meaning of the whole. In practice, the same letter can represent several separate bound variables (with separate boxes), that can take different values without conflict, as no two of them are anywhere free together to compare their values. The common language does this continuously, using very few variable symbols ("he", "she", "it"...)

Functions

A *function* is any object f behaving as a variable whose value is determined by that of another variable called its *argument*, which has a range called the *domain* of the function and denoted $\operatorname{Dom} f$. Whenever its argument is fixed (gets a value and a name, say x), f becomes a constant (denoted $f(x)$). In other words, f is made of the following data:

- A set called the *domain* of f, denoted $\operatorname{Dom} f$
- For each element x of $\operatorname{Dom} f$, an object $f(x)$ called the *image of x by f* or *value of f at x*.

Operations

The notion of *operation* generalizes that of function, by admitting a finite list of arguments (variables with given respective ranges) instead of one. So it gives a result (a value) when all its arguments are fixed. The number n of arguments of an operation is called its *arity*; the operation is called *n-ary*. It is called *unary* if $n = 1$ (it is a function), *binary* if $n = 2$, *ternary* if $n = 3$... Nullary operations are useless as their role is played by their unique value; we shall see how to construct those of arity > 1 by means of functions.

The value of a binary operation f on its fixed arguments named x and y (i.e. its value when its arguments are assigned the values of x and y), is denoted $f(x, y)$. So instead of symbols, the arguments are represented by the left and right spaces in parenthesis, to be filled by any expression giving them desired values.

1.3. Form of theories: notions, objects and meta-objects

Notions and objects

Each theory has its own list of *notions*, usually designated by common names, that are the kinds of variables used by the theory ; each model (interpretation of the theory) inteprets each notion as a set that is the common range of all variables of this kind. For example, Euclidean geometry has the notions of "point", "straight line", "circle" and more. The *objects* of a theory in a model, are all the possible values of its variables (the elements of its notions) in this model.

One-model theory

When we discuss several theories T and systems M that are models of T, we are in the framework of model theory, with the notions of "theory" and "system" that are the respective kinds of the variables T and M. But when we focus the study on one theory (such as set theory) with a fixed model, the variables T and M become fixed and disappear (they are not variables anymore, the choice of the theory and the model becomes implicit). So, the notions of theory and model disappear too. This fixation reduces the framework, from model theory, to that of *one-model theory*.

A model of one-model theory, is a system $[T, M]$ that combines a theory T with a model M of T.

On the diversity of logical frameworks

Before giving a theory T, we must specify its *logical framework* (its format, or grammar), by which the contents of T can be expressed, and their consequences can be deduced. This framework is given by the choice of the version of one-model theory that describes T and interprets its claims.

We shall first describe 2 logical frameworks in parallel. Theories in the most common framework of *first-order logic*, will be called here *generic theories*. Set theory will be expressed in is own special framework. More frameworks will be introduced in Part 3.

The most common logical frameworks except that of our set theory, will manage notions as *types* (usually in finite number) classifying both variables and objects: each object will belong to only one type, the one of variables that can name it. For example, an object of Euclidean geometry may be either a point or a straight line, but the same object cannot be both a point and a straight line.

(*) http://en.wikipedia.org/wiki/Irresistible_force_paradox

Examples of notions from various theories

Theory	Kinds of objects (notions)
Generic theory	Pure elements classified by types
Set theory	Elements, sets, functions, operations, relations, tuples...
Model theory	Theories, systems and their components (listed next line)
One-model theory	Objects, symbols, types, structures, expressions (terms, formulas)...
Arithmetic	Natural numbers
Linear Algebra	Vectors, scalars...
Geometry	Points, staight lines, circles...

Meta-objects

The notions of a one-model theory T_1, normally interpreted in $[T, M]$, classify the components of T ("type", "symbol", "formula"...), and those of M ("object", and tools to interpret T there). But we can interpret the same notions in $[T_1, [T, M]]$, by putting the prefix meta- on them.

By its notion of "object", one-model theory distinguishes the objects of T in M among its own objects in $[T, M]$, that are the meta-objects. The above rule of use of the meta prefix would let every object be a meta-object; but we will make a vocabulary exception by only calling *meta-object* those which are not objects: symbols, types (and other notions), structures, expressions...

Set theory only knows the ranges of some of its own variables, seen as objects (sets). But every variable of any theory has a range among notions, which are meta-objects. notions themselves)

Components of theories

Once chosen a framework, the content of a theory (or its foundation, i.e. its initial content, describing its intended kind of models), consists in a choice of 3 successive lists of components, where those in each list are used to build those of the next list:

• A list of *abstract types* that will name the types of objects;

• A *language* (vocabulary): list of *structure symbols*, names of structures that will relate objects of given types to form systems (see 1.4).

• A list of *axioms* (among closed formulas of the theory, see 1.5.).

Set-theoretical interpretation

Any generic theory T can be inserted into set theory by converting its components into components of set theory, as follows. All abstract types and structure symbols of T become fixed variables (or new constant symbols) whose values are the main components of its model (itself an object of set theory), that determine it (and vary with it). This integrates all theories as parts of the same set theory, and all their possible models as parts of a common model of set theory (also called *universe*).

The value of each abstract type is a set called an *interpreted type* (the range of variables of that type). For geometry, both abstract types "Point" and "Straight line" become fixed variables P and L, respectively designating the set of all points, and the set of all straight lines.

Variables of T remain variables, but all their values (objects) become pure elements. This method works for any generic theory, though some theories like geometry are usually treated differently (interpreting straight lines as sets of points).

1.4. Mathematical structures

The *structures*, meant by structure symbols (their values in set theory), will give the studied system its shape by relating objects of different types, giving them roles which can sometimes be interpreted as that of complex objects (despite being pure elements for set theory). Generic theories admit 2 kinds of structures (and thus of structure symbols): operators and predicates.

An *operator* is an operation between interpreted types. Before interpretation, the theory specifies for each operator symbol its arity (or list of arguments seen as places around the symbol), the type of each argument (which will range over the interpreted type), and the (common) type of all its values (results). The *constant symbols* of a theory are its nullary operator symbols. A unary operator (that is a function) will be called a *functor*.

The list of types is completed by the *Boolean* type, interpreted as the pair of elements 1 ("true") and 0 ("false"). A variable of this type (outside the theory) is called a *Boolean variable*. A *para-operator* is a generalized operator allowing the Boolean type among its types of arguments and results. A *connective* is a para-operator with only Boolean arguments and values. A *predicate* is a para-operator with one or more arguments but no Boolean one, and with Boolean values.

Structures of set theory

Let us start formalizing set theory with 3 primitive notions : elements (i.e. all objects), sets and functions. This will be progressively developed with other notions that may be seen either as primitive or derived from the former, and other symbols contributing to give their roles to all kinds of objects. But we first need to ignore the above set theoretical interpretation of theories, whose use of set theory should not be confused with the one we want to formalize.

Sets E play their roles by the binary predicate $\in$: for any element x, we say that x is in E (or is an *element of* E) and write $x \in E$, to mean that x is a possible value of a variable with range E.

Functions f play their role by two operators: the domain functor Dom, and the *function evaluator*, binary operator implicitly written as $f(x)$ with arguments f and x, giving the value of f at any $x \in \operatorname{Dom} f$.

About ZFC set theory

The ZFC axiomatic set theory (Zermelo-Fraenkel with the axiom of choice) is a generic theory with only one type "set", one structure symbol $\in$, and axioms. Thus, it assumes that every object is a set, and thus a set of sets and so on, built over the empty set. Specialists of mathematical logic chose it for their need of a powerful theory in an enlarged founding cycle, by which they can prove many difficult formulas or their unprovability. Then, authors of basic courses usually saw set theory as a popularized or implicit version of ZFC, considered the standard reference, as if this was necessary or obvious (as intuitions historically selected for their consistency and convenience).

But for a start of mathematics, ZFC is not an ideal reference. Its axioms (descriptions of the universe that must assume the framework of model theory to make sense) would deserve more subtle and complex justifications than usually assumed. Ordinary mathematics, using many objects not seen as sets, is rather awkward to develop from it. As the roles of all needed objects can be played by sets, they did not require another formalization, but remained a discrepancy between the "theory" and the practice of mathematics.

Types in one-model theory

One-model theory only needs one meta-notion of "type" to play both roles of abstract types (in the theory) and of their values as interpreted types (components of the model) : these roles are given by meta-functors, one from variables to types, and one from objects to types. This way, one-model theory ignores the more general notion of "set of objects" by which we introduced interpreted types.

The notion of structure in one-model theory

Similarly, the role of "structures" (that is an operation between interpreted types, with Boolean values if a predicate) can be formalized by meta-structures (similar to the function evaluator). Unlike interpreted types all named by abstract types, our notion of structure in one-model theory will be larger than those named by the symbols in the language. For this, structures will be another meta-type with a meta-functor from symbols to structures.

However, this formalization would leave the exact range of this notion of structure undetermined. Trying to conceive this range as that of "all operations between interpreted types" would beg the question of where the knowledge of this totality may come from. This idea of totality will be formalized as the powerset in set theory (2.5.), but its meaning will still depend on the universe (model) of set theory where it is interpreted, far from our present concern for one-model theory.

Instead, we will choose the notion of structure to mean those definable by expressions.

1.5. Expressions and definable structures

Terms and formulas

Let us sketch the description of expressions in theories (that will be formalized much later).

An *occurrence* of a symbol in an expression is a place where it is written, for example "$x + x$" has two occurrences of x and one of $+$. Given the first two layers of a theory (a list of types and a language), an *expression* (either a term or a formula) is a finite system of occurrences of symbols, that will give a value for each choice of a system interpreting types and structure symbols, with values of free variables in it. The values of *terms* are objects, while *formulas* have boolean values.

Expressions can use the following symbols :

- Variables of each type ;
- Structure symbols (from the language) ;
- One equality symbol per type (predicate with 2 arguments of this type) abusively all denoted $=$, interpreted in the standard way ;

• Connectives ;

• Binders: both quantifiers (∀ and ∃, see 1.9.) are the only binders in generic theories, but our set theory will have more (see 1.8.).

Expressions are built successively. The first and simplest ones are made of just one symbol already having a value by itself: symbols of constant or variable, called *atomic terms* ; the Boolean constants 1 and 0 are the first formulas. Other expressions are made of a distinguished choice (occurrence) of a symbol called its *root*, and a list of entries chosen among previously built expressions, plus a choice of one (or more) variable symbol(s) in the case of binders. The root determines the type of values (thus decides if it is a formula or a term) and the format of the list (the number and types of entries). For para-operator symbols, this format is given by the list of arguments.

The list was empty for constants and variables, that (as root) formed expressions alone. Other symbols need a nonempty list, which requires a choice of display convention. Most symbols of binary para-operators are displayed between their arguments (such as $x + y$), while a function-like display as $+(x, y)$ can better suit other arities. Some others (multiplication, exponentiation) may appear implicitly, without character. Some symbols are denoted by several characters delimiting the entries. The parenthesis can either be part of the notation of a symbol (function evaluator, tuples...), or be used to separate the subexpressions and distinguish the root, as in $(x + y)^n$.

An expression is *closed*, if it has no free variable (all the variables in it are subject to binders, see 1.8) so that its value only depends on the system interpreting the given types and structure symbols. The list of axioms of a theory is chosen among its closed formulas.

About undefined structures

Confusing a nullary operation with its value, we find "all nullary operators", to just be all objects. So, we can reach all nullary operators of each type, by using and binding a variable of the theory. Variables can be used in expressions but are not "in the language" of the theory; those in the language (symbols with the same shape to play the same role but with a specific name for one specific value), are the constants.

A structure with nonzero arity not named in the language, might be named by a new structure symbol, but this escapes the above list of accepted symbols in expressions. Thus, expressions using this symbol need a new status: either an extended framework accepting variable structure symbols (such as second-order logic, see 3.6), or an extended theory where this symbol is added to the language (this choice of interpretation generalizes to other arities, the choice between seeing a symbol as a variable or a constant depending on the theory we consider to be in).

As a theory cannot handle the full range of a variable structure with nonzero arity (unless all its arguments range over finite sets), not all properties of this range can be known (nor even formally expressed), but some can. Namely, if a formula with a variable structure symbol is formally proven, then it is true for all structures it might mean, no matter which of these structures "can be found" or not. This notion of provable universal truth will be used as a rule of proof for ordinary variables (principle of Universal Introduction in 1.9) and in the expression of the set generation principle (1.11).

Structures defined by expressions

Any theory can produce (discover) structures (operators or predicates) beyond those directly named in its language, as the operation between interpreted types defined by the following data:

• An expression, some of whose free variables (possibly unused) are the arguments of the intended structure, that will be bound by this definition ;

• All other used free variables, called *parameters*, get a choice of fixed values.

For example, in a formalization of set theory, a function f is synonymous with the functor defined by the term "$f(x)$" with argument x and parameter f.

We can accept as a legitimate work inside the theory, to name such a structure by a symbol of variable structure, meant as an abbreviation of the expression with its list of parameters. This variable structure can range over all structures defined by a common expression with all possible values of its parameters: the variability of the symbol abbreviates those of parameters, and thus can be bound in the theory by binding the parameters on the defining expression.

The variable nullary operator ranging over a given type, is defined by the atomic term made of a variable of this type seen as parameter.

We can fix the notion of structure in one-model theory, as made of all structures that can be reached this way, defined by any expression with any values of parameters. As this involves the infinite range of all expressions, it escapes the abilities of the studied theory (which can only use one expression at a time) and is only accessible by the framework of one-model theory (with its meta-notion of expression). Still this generality does not mean absolute exhaustiveness (there may be undefinable structures).

Invariant structures

An *invariant structure* is a structure defined without parameters, thus a constant one. The distinction of invariant structures from other structures, generalizes the distinction between constants and variables, both to cases of nonzero arity, and to what is indirectly expressible by the theory instead of directly named in its language. All structures named in the language of a theory are invariant for it (directly defined by its symbols). Other invariant structures can be named by new symbols to be added to the language of a theory, preserving its deep meaning (provability, structures, invariant structures. . .). Such rules to develop a theory will be discussed in 3.2.

1.6. Connectives

The nullary connectives are 0 and 1. The binary connective of equality between Booleans is written $\Leftrightarrow$ and called *equivalence*. Here are the most useful connectives ordered by arity $n > 0$, with their properties for all values of the Boolean variables (abbreviations of formulas) A, B, C.

$n = 1$: the *negation* $\neg$ exchanges Booleans ($\neg A$ is read "not A"): $\neg 1 \Leftrightarrow 0, \neg 0 \Leftrightarrow 1, \neg(\neg A) \Leftrightarrow A$. Also denoted by barring the root of its argument (forming another symbol with the same format): $(x \neq y) \Leftrightarrow \neg(x = y)$ ("x is different from y"); $x \notin E \Leftrightarrow \neg(x \in E)$ ("x does not belong to E").

$n = 2$:

$\wedge$ (*conjunction*) means "and", giving true only when both arguments are true;

$\vee$ (*disjunction*) means "or", giving true except when both arguments are false;

$\Rightarrow$ (*implication*): $A \Rightarrow B$ can be read "A implies B", "A is a sufficient condition for B", or "B is a necessary condition for A", and means $((\neg A) \vee B)$. It is true except when A is true and B is false, thus expresses that if A is true then B also, but gives no information otherwise (being true).

The formula $(\neg B \Rightarrow \neg A)$ is called the *contrapositive* of $(A \Rightarrow B)$, and is equivalent to it.

$\Leftrightarrow$: $(A \Leftrightarrow B) \Leftrightarrow ((A \Rightarrow B) \wedge (B \Rightarrow A))$. A proof of $A \Leftrightarrow B$ can be made of a proof of the implication $(A \Rightarrow B)$, then a proof of the second one $(B \Rightarrow A)$, called the *converse* of $(A \Rightarrow B)$.

Its negation $A \not\Leftrightarrow B$ can also be read as $(A \Leftrightarrow \neg B)$, or as "exclusive or" $((A \vee B) \wedge \neg(A \wedge B))$.

Negations exchange various connectives:

$$\begin{aligned}(A \vee B) &\not\Leftrightarrow (\neg A \wedge \neg B)\\ (A \wedge B) &\not\Leftrightarrow (\neg A \vee \neg B)\\ (A \Rightarrow B) &\not\Leftrightarrow (A \wedge \neg B)\end{aligned}$$

This transforms the properties of associativity and distributivity into various formulas:

$$\begin{aligned}((A \wedge B) \wedge C) &\Leftrightarrow (A \wedge (B \wedge C))\\ ((A \vee B) \vee C) &\Leftrightarrow (A \vee (B \vee C))\\ (A \Rightarrow (B \Rightarrow C)) &\Leftrightarrow ((A \wedge B) \Rightarrow C)\\ (A \Rightarrow (B \vee C)) &\Leftrightarrow ((A \Rightarrow B) \vee C)\end{aligned}$$

$$\begin{aligned}(A \wedge (B \vee C)) &\Leftrightarrow ((A \wedge B) \vee (A \wedge C))\\ (A \vee (B \wedge C)) &\Leftrightarrow ((A \vee B) \wedge (A \vee C))\\ (A \Rightarrow (B \wedge C)) &\Leftrightarrow ((A \Rightarrow B) \wedge (A \Rightarrow C))\\ ((A \vee B) \Rightarrow C) &\Leftrightarrow ((A \Rightarrow C) \wedge (B \Rightarrow C))\\ ((A \Rightarrow B) \Rightarrow C) &\Leftrightarrow ((A \vee C) \wedge (B \Rightarrow C))\\ (A \wedge (B \Rightarrow C)) &\Leftrightarrow ((A \Rightarrow B) \Rightarrow (A \wedge C))\end{aligned}$$

$n \geq 3$: strings of conjunctions such as $(A \wedge B \wedge C)$, abbreviate formulas with more parenthesis as in $((A \wedge B) \wedge C)$, whose deletion is allowed by associativity ; and similarly for strings of disjunctions such as $(A \vee B \vee C)$. Asserting a conjunction of formulas amounts to asserting all these formulas.

In a different kind of abbreviation, any string of formulas linked by $\Leftrightarrow$ and/or $\Rightarrow$ will mean the conjunction of all these implications or equivalences between adjacent formulas:

$$0 \Rightarrow A \Rightarrow A \Rightarrow 1$$
$$(\neg A) \Leftrightarrow (A \Rightarrow 0) \Leftrightarrow (A \Leftrightarrow 0)$$
$$(A \Rightarrow B \Rightarrow C) \Leftrightarrow ((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$$
$$(A \Leftrightarrow B \Leftrightarrow C) \Leftrightarrow ((A \Leftrightarrow B) \wedge (B \Leftrightarrow C)) \Rightarrow (A \Leftrightarrow C)$$
$$(A \wedge A) \Leftrightarrow (A \wedge 1) \Leftrightarrow A \Leftrightarrow (A \vee A) \Leftrightarrow (A \vee 0) \Leftrightarrow (1 \Rightarrow A) \Leftrightarrow (A \Leftrightarrow 1)$$
$$(B \wedge A) \Leftrightarrow (A \wedge B) \Leftrightarrow (A \wedge (A \Rightarrow B)) \Rightarrow B \Rightarrow (A \vee B) \Leftrightarrow (B \vee A)$$

Axioms of equality

For any objects (or abbreviations of terms) x, y, any functor T and any unary predicate A,

$$x = y \Rightarrow T(x) = T(y)$$
$$x = y \Rightarrow (A(x) \Leftrightarrow A(y))$$

Thus an equality between terms $x = y$ allows to replace any occurrence of x by y in any expression without affecting the result. In particular when a symbol is defined by a term, both are equal, thus can be substituted to each other in any expression. Axioms and other rules expressed with variable symbols (under universal quantifiers, see 1.9) can then be used replacing these variables by terms.

1.7. Classes in set theory

Set theory in the founding cycle

Attempts to formalize one-model theory in first-order logic, would fail to exclude infinitely large "expressions" and "proofs": this needs a second-order axiom, expressible after insertion into set theory (as we shall see in Part 3). As the components of its model $[T, M]$ are named there by free variables, their variability makes this the set-theoretical expression of model theory (that, together with the axiomatization of set theory, will complete the grand tour of the foundations of mathematics).

Now let T_0 be the external copy of T, i.e. the theory (inserted in set theory) made of components k (types, symbols, axioms) whose copies as objects are (truly finite) components of T (formally, "k"$\in T$ where the quotation "k" abbreviates a closed term of set theory describing k as an object). By this correspondence, any (model-theoretical) model M of T is also a (set-theoretical) model of T_0.

This context gives to the interpretation of T_0 in M, a powerful framework encompassing both (set-theoretical and model-theoretical) frameworks of interpretations of theories in models. Namely, all works (expressions, developments) we do in T_0, have copies as objects in the system T described by set theory as a system of objects that "is a theory"; while the (set theoretical) model M of T_0 is formally seen as a "model of T", in the model-theoretical sense formalized inside the same set-theoretical framework, as it belongs to the same universe (model of set theory) as T.

However, the power of this interpretation comes with a cost in legitimacy: given a consistent theory T_0, the existence of a corresponding T with a model M is not automatic. If T_0 has infinitely many components, they must be produced by some rules, for getting T as defined by the same rules (infinite lists cannot be written). Then, attempts to justify the existence of M (tied with the formal consistency of T), would face obstacles as shown by the incompleteness theorem (3.9).

But we shall now formalize set theory as a theory T (that can take the form of a generic theory as explained in 1.9 and 1.10), and thus interpret set theoretical concepts in the model M. Thus, our above use of set theoretical concepts, identical to T_0 but interpreted as describing the surrounding universe, will need to be distinguished from it by carrying the meta- prefix. Set theoretical concepts in M can be nicely reflected by their meta interpretation, but both should not be confused.

Classes

For any theory, we can define a *class* as a unary predicate seen as the meta-set of objects where it is true. In particular for set theory, each set E is synonymous with the class defined by the formula "$x \in E$" with argument x and parameter E, which we read as "the class of the x such that $x \in E$". Instead of the standard representation of all objects of generic theories as pure meta-elements, the role of the objects "sets" of set theory will usually be played by meta-sets of the same elements (and similarly for functions), letting any set E be a class, while any class is a meta-set of objects. But some meta-sets of objects are not classes (i.e. are not definable by any formula); and Russell's paradox (in 1.8) will show that some classes are not sets (such as the class of all objects, defined by 1, or the class of all sets).

Definiteness classes

In set theory, all objects need to be included as "elements" that belong to sets and can be operated by functions (to avoid illimited divisions between sets of elements, sets of sets, sets of functions...). This might be formalized by keeping 3 types where each set would have a copy among elements (by a functor from sets to elements), and the same for functions. But it would not suffice to our set theory: beyond sets and functions, we will need more notions, that will be better expressed as classes than as types. The notions of sets and functions will be classes named by symbols: Set = "is a set", Fnc = "is a function".

In generic theories, the syntactic correction of an expression (that is implicit in the concept of "expression") ensures that it will take a definite value, for every data of a model with a fixed system of values of its free variables in this model. But in set theory, this may still depend on the values of its free variables. So, a structure (or expression) $\mathcal{A}$ will be called *definite* when it actually takes a value for the given values of its arguments (or free variables) in the model. This definiteness condition of $\mathcal{A}$ is itself a definite predicate (expressed by a formula) $d\mathcal{A}$, with the same arguments. Classes are defined by everywhere definite unary predicates. The meta-domain of any unary structure $\mathcal{A}$, is the class defined by $d\mathcal{A}$, with the same argument and parameters, called its *class of definiteness.*

Expressions should be only used where they are definite, which will be done rather naturally. The definiteness condition of $(x \in E)$ is $\mathrm{Set}(E)$. That of the function evaluator $f(x)$ is $(\mathrm{Fnc}(f) \wedge x \in \mathrm{Dom}\, f)$. But the definiteness of the last formula must be explained as follows.

Extended definiteness

A theory with partially definite structures can be formalized (translated) as a theory with one type and everywhere definite structures, keeping intact all expressions and their values wherever they are definite : models are translated one way by giving arbitrary values to indefinite structures (e.g. a constant value), and in the way back by ignoring those values. Thus, an expression with an indefinite subexpression may be declared definite if its value does not depend on these extra values.

For all predicates $\mathcal{A}$ and $\mathcal{B}$, let us give to $\mathcal{A} \wedge \mathcal{B}$ and $\mathcal{A} \Rightarrow \mathcal{B}$ the same definiteness condition $(d\mathcal{A} \wedge (\mathcal{A} \Rightarrow d\mathcal{B}))$ (breaking, for $\mathcal{A} \wedge \mathcal{B}$, the symmetry between $\mathcal{A}$ and $\mathcal{B}$, that needs not be restored). They will thus be seen definite if $\mathcal{A}$ is false and $\mathcal{B}$ is not definite, with respective values 0 and 1.

This ensures the definiteness of the definiteness conditions themselves, as well as of $d\mathcal{A} \wedge \mathcal{A}$ and $d\mathcal{A} \Rightarrow \mathcal{A}$ (respectively extending $\mathcal{A}$ by 0 and 1 where it was not definite). Formulas $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$ and $(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}$ have the same definiteness condition $(d\mathcal{A} \wedge (\mathcal{A} \Rightarrow (d\mathcal{B} \wedge (\mathcal{B} \Rightarrow d\mathcal{C}))))$.

For any class $\mathcal{A}$ and any unary predicate $\mathcal{B}$ definite in all $\mathcal{A}$, the class defined by the (everywhere definite) predicate $(\mathcal{A} \wedge \mathcal{B})$, is called the *subclass of $\mathcal{A}$ defined by $\mathcal{B}$.*

1.8. Bound variables in set theory

The syntax of binders

The last kind of symbol forming expressions is the binders (binding symbols), which bind one (or more) variable(s) (say x) on an expression F which may use x as a free variable in addition to the free variables that are available (with a value) outside. Such a symbol thus separates the "inside" subexpression F using x as free, from the "outside" where x is bound. Applied to the data of the symbol x and the expression F, it gives a value depending on the unary structure defined by F with argument x (this unary structure would have "too much information" to be fully recorded as an object). A *quantifier* is a binder giving a Boolean value from a unary predicate.

An expression is *closed* if all its variables are bound (contained by binders).

The precise syntax differs between set theory and generic theories, which manage the range of x differently. In generic theories, ranges are types, implicit data of quantifiers (the only binders there). But binders of set theory make their variable x range over a set, that is an object to be given as the value of an extra argument (a space to insert a term not using x and designating a set), in addition to the above data.

Let us review the main binders in set theory.

Definitions of functions by terms

The *function definer* ($\ni \mapsto$) binds a variable x with range E on a term here abbreviated as $t(x)$ (with implicit parameters), following the syntax $(E \ni x \mapsto t(x))$, sometimes shortened as $(x \mapsto t(x))$ when E is determined by the context. Definite if $t(x)$ is definite for all x in E, it takes the functor defined by t and restricts its definiteness class to E, to give it as a function with domain E.

So it converts functors into functions, reversing the action of the function evaluator (that converted functions into their roles as functors whose definiteness classes were sets), while functors were otherwise only indirectly available to be used as functions, by means of their defining term that may have unlimited complexity. In 1.10. we will formalize this tool and see it as a particular case of a more general principle for set theory.

Relations and set-builder symbol

A *relation* is like an operation but with Boolean values, acting as a predicate whose arguments range over sets.

For any unary predicate $\mathcal{R}$ definite in a set E, the subclass of E defined by $\mathcal{R}$ is a set (range of a variable x introduced as ranging over E, so that it can be bound, from which we select the cases satisfying $\mathcal{R}(x)$), thus a *subset of* E, denoted $\{x \in E|\mathcal{R}(x)\}$ (set of x in E such that $\mathcal{R}(x)$): for all y,

$$y \in \{x \in E|\mathcal{R}(x)\} \Leftrightarrow (y \in E \wedge \mathcal{R}(y))$$

The *set-builder* $\{\ \in\ |\ \}$, binding x to E on $\mathcal{R}$, will be used as a definer for unary relations on E, figured as subsets F of E, evaluated by $\in$ as predicates $(x \in F)$ with argument x. But these predicates are definite throughout the universe, giving 0 outside E whose data is lost. This lack of operator Dom does not matter, as the domain E is usually determined by the context.

As the function definer (resp. the set-builder) records the whole structure defined by the given expression on the given set, it suffices to define any other binder on the same expression with the same domain, as made of a functor or unary predicate applied to its result (that is a function, resp. a set).

1.9. Quantifiers

In set theory, the full syntax for a quantifier Q binding a variable x with range E on a unary predicate $\mathcal{R}$ (i.e. a formula abbreviated as $\mathcal{R}(x)$), will be $Qx \in E, \mathcal{R}(x)$. Otherwise, the domain (type or class) may be kept implicit (if fixed by the context) by the notation $Qx, \mathcal{R}(x)$, or displayed as an index : $Q_E x, \mathcal{R}(x)$. The two main quantifiers (by which others will be defined later) are:

- The *existential quantifier* $\exists$, that reads “There exists x (in…) such that…”
- The *universal quantifier* $\forall$, that reads “For all (or: for any) x (in…),…”).

They can be meta-defined using the meta-function $(x \mapsto \mathcal{R}(x))$ with the same domain, by

$$\begin{gathered}(\forall x, \mathcal{R}(x)) \ \Leftrightarrow\ (x \mapsto \mathcal{R}(x)) = (x \mapsto 1)\\ (\exists x, \mathcal{R}(x)) \Leftrightarrow\ (x \mapsto \mathcal{R}(x)) \neq (x \mapsto 0)\\ (\forall x, \mathcal{R}(x)) \ \not\Leftrightarrow\ (\exists x, \neg\mathcal{R}(x))\end{gathered}$$

In set theory, $(\forall x \in E, \mathcal{R}(x)) \Leftrightarrow \{x \in E|\mathcal{R}(x)\} = E$. The formula $(\forall x, 1)$ is always true. With classes,

$$\begin{gathered}(\exists_{\mathcal{C}} x, \mathcal{R}(x)) \Leftrightarrow (\exists x, \mathcal{C}(x) \wedge \mathcal{R}(x)) \Leftrightarrow \exists_{\mathcal{C}\wedge\mathcal{R}} x, 1\\ (\forall_{\mathcal{C}} x, \mathcal{R}(x)) \Leftrightarrow (\forall x, \mathcal{C}(x) \Rightarrow \mathcal{R}(x))\\ \forall x, (\mathcal{C}(x) \Leftrightarrow (\exists_{\mathcal{C}} y, x = y))\end{gathered}$$

Inclusion between classes

A class $\mathcal{A}$ is said to be included in a class $\mathcal{B}$ when $\forall x, \mathcal{A}(x) \Rightarrow \mathcal{B}(x)$. Then $\mathcal{A}$ is a subclass of $\mathcal{B}$, as $\forall x, \mathcal{A}(x) \Leftrightarrow (\mathcal{B}(x) \wedge \mathcal{A}(x))$. Conversely, any subclass of $\mathcal{B}$ is included in $\mathcal{B}$. The inclusion of $\mathcal{A}$ in $\mathcal{B}$ implies for any predicate $\mathcal{C}$ (in cases of definiteness):

$$\begin{gathered}(\forall_{\mathcal{B}} x, \mathcal{C}(x)) \Rightarrow (\forall_{\mathcal{A}} x, \mathcal{C}(x))\\ (\exists_{\mathcal{A}} x, \mathcal{C}(x)) \Rightarrow (\exists_{\mathcal{B}} x, \mathcal{C}(x))\\ (\exists_{\mathcal{C}} x, \mathcal{A}(x)) \Rightarrow (\exists_{\mathcal{C}} x, \mathcal{B}(x))\\ (\forall_{\mathcal{C}} x, \mathcal{A}(x)) \Rightarrow (\forall_{\mathcal{C}} x, \mathcal{B}(x))\end{gathered}$$

Rules of proofs for quantifiers on a unary predicate $\mathcal{R}$

Existential Introduction. *If we have terms $t, t', \cdots$ and a proof of $\mathcal{R}(t) \vee \mathcal{R}(t') \vee \cdots$, then $\exists x, \mathcal{R}(x)$.*

Existential Elimination. *If $\exists x, \mathcal{R}(x)$, then we can introduce a new free variable z with the hypothesis $\mathcal{R}(z)$ (the consequences will be true without restricting the generality).*

These rules express the meaning of $\exists$: going from $t, \cdots$ to $\exists$ then from $\exists$ to z, is like letting z represent one of $t, t', \cdots$ (without knowing which). They give the same meaning to $\exists_{\mathcal{C}}$ as to its 2 above equivalent formulas, bypassing the extended definiteness rule for $(\mathcal{C} \wedge \mathcal{R})$ by focusing on the case when $\mathcal{C}(x)$ is true and thus $\mathcal{R}(x)$ is definite.

While $\exists$ appeared as the designation of an object, $\forall$ appears as a deduction rule:

Universal Introduction. *If from the mere hypothesis $\mathcal{C}(x)$ on a new free variable x we could deduce $\mathcal{R}(x)$, then $\forall_{\mathcal{C}} x, \mathcal{R}(x)$.*

Universal Elimination. *If $\forall_{\mathcal{C}} x, \mathcal{R}(x)$ and t is a term satisfying $\mathcal{C}(t)$, then $\mathcal{R}(t)$.*

Introducing then eliminating $\forall$ is like replacing x by t in the initial proof.

Deductions can be made by these rules, reflecting formulas

$$((\forall_{\mathcal{C}} x, \mathcal{A}(x)) \wedge (\forall_{\mathcal{C}} x, \mathcal{A}(x) \Rightarrow \mathcal{B}(x))) \Rightarrow (\forall_{\mathcal{C}} x, \mathcal{B}(x))$$
$$((\exists_{\mathcal{C}} x, \mathcal{A}(x)) \wedge (\forall_{\mathcal{C}} x, \mathcal{A}(x) \Rightarrow \mathcal{B}(x))) \Rightarrow (\exists_{\mathcal{C}} x, \mathcal{B}(x))$$
$$(\forall_{\mathcal{C}} x, \mathcal{A}(x)) \wedge (\exists_{\mathcal{C}} x, \mathcal{B}(x))) \Rightarrow (\exists_{\mathcal{C}} x, \mathcal{A}(x) \wedge \mathcal{B}(x))$$
$$(\exists_{\mathcal{A}} x, \forall_{\mathcal{B}} y, \mathcal{R}(x,y)) \Rightarrow (\forall_{\mathcal{B}} y, \exists_{\mathcal{A}} x, \mathcal{R}(x,y))$$

Status of open quantifiers in set theory

Set theory is translated to a generic theory by converting to classes the domains of quantifiers:

$$(\exists x \in E, \mathcal{R}(x)) \to (\exists x, x \in E \wedge \mathcal{R}(x))$$
$$(\forall x \in E, \mathcal{R}(x)) \to (\forall x, x \in E \Rightarrow \mathcal{R}(x))$$

Set theory only admits quantifiers over sets, called *bounded quantifiers*, in its formulas (also called *bounded formulas* for insistence) that define predicates and can be used in terms. But its translated form as a generic theory allows quantifiers on classes (or the universe), called *open quantifiers*. Formulas with open quantifiers in set theory will be called *claims*. Their use will be essentially restricted to declarations of truth of closed definite claims. These will first be axioms, then theorems (deduced from axioms, and differently named according to their importance: a *theorem* is more important than a *proposition*, may be deduced from a *lemma*, and a *corollary* may be easily deduced from it).

Open quantifiers in claims will usually be expressed as common language articulations (implicitly using the above rules of proofs) between their bounded sub-formulas (written in set-theoretic symbols).

The set-builder was defined by a claim but 1.11 will redefine it without open quantifier (except on its parameters). We will now use it to show that the class of all sets is not a set, which will oblige to keep all these distinctions between classes and sets, and between open and bounded quantifiers:

Theorem (Russell's paradox). *For any set E there is a set F such that $F \notin E$.*

Proof. $F = \{x \in E | \operatorname{Set}(x) \wedge x \notin x\} \Rightarrow (F \in F \Leftrightarrow (F \in E \wedge F \notin F)) \Leftrightarrow (F \notin F \wedge F \notin E)$. □

1.10. Formalization of set theory

The inclusion predicate

The inclusion predicate $\subset$ between two sets E and F, is defined by $E \subset F \Leftrightarrow (\forall x \in E, x \in F)$. It reads: E is included in F, or E is a subset of F, or F includes E.

We always have $E \subset E$. Implications chains also appear as inclusion chains:

$$(E \subset F \subset G) \Leftrightarrow (E \subset F \wedge F \subset G) \Rightarrow E \subset G.$$

First axioms

$$\begin{aligned} &\forall x, \quad \neg(\operatorname{Set}(x) \wedge \operatorname{Fnc}(x)) \\ &\forall_{\operatorname{Fnc}} x, \quad \operatorname{Set}(\operatorname{Dom} x) \\ \text{(For any term } t\text{)}, \; &\forall E, \forall(\text{parameters}), \operatorname{Fnc}(E \ni x \mapsto t(x)) \\ &\forall_{\operatorname{Set}} E, F, \quad E \subset F \subset E \;\Rightarrow E = F (\text{Axiom of Extensionality}). \end{aligned}$$

The latter redefines equality between sets by their equivalence as classes (letting elements in bulk): $E \subset F \subset E$ means that E and F have the same elements ($\forall x, x \in E \Leftrightarrow x \in F$) and implies for any predicate $\mathcal{R}$ that $(\forall x \in F, \mathcal{R}(x)) \Leftrightarrow (\forall x \in E, \mathcal{R}(x))$, and similarly for $\exists$.

Translating the definer

When translating set theory as a generic theory, the function definer becomes an infinity of operator symbols: for each term t with one argument (and parameters), the whole expression $(E \ni x \mapsto t(x))$ becomes another operator symbol whose arguments are E and the parameters of t. (Those where every subexpression inside t without any occurrence of x is the only occurrence of a parameter, suffice to define others).

The following axioms can be read as axioms of the generic theory into which set theory is converted; those depending on a term t are schemas of axioms (schema of claims = infinite list of claims obtained by replacing an extra structure symbol by any possible defining expression):

Axioms for functions. *For any functor t (term with one argument), any values of its parameters, any set E over which t is definite, and any functions f and g, the first of these axioms summarizes the next 3 ones:*

$$f = (E \ni x \mapsto t(x)) \Leftrightarrow (\mathrm{Dom}\, f = E \wedge (\forall x \in E, f(x) = t(x)))$$
$$\mathrm{Dom}(E \ni x \mapsto t(x)) = E$$
$$\forall x \in E, (E \ni y \mapsto t(y))(x) = t(x)$$
$$(\mathrm{Dom}\, f = \mathrm{Dom}\, g \wedge \forall x \in \mathrm{Dom}\, f, f(x) = g(x)) \Rightarrow f = g$$

A general principle for the formalization of set theory

For any kind of meta-objects indirectly expressible and usable like objects in expressions, set theory will be enriched with tools to directly present them as objects. Namely, classes behaving as sets will become sets (1.11); indirectly specified elements will become directly specified (2.4); the functor of cardinality can be legitimately accepted. But when the indirect expression of meta-objects (here, functors) was defined by any of an infinite list of possible expressions (here, any term), their gathering into a single kind of clearly definite objects (here, functions), needs to be justified by another reason (here, the restriction of their domain to a set).

For cases of meta-objects behaving as objects of a kind that is neither sets nor functions, this new kind of objects will be introduced (operations, relations, tuples), and the conversion tools from roles (meta-objects) to objects, will be completed by new conversion tools from objects into their roles. But this can be done inside the same set theory (just by developing it), as already present objects (sets or functions) can be found to play the roles of these new objects. So, the new notions can be defined as classes of existing objects (that will offer their expressible features to the new objects), while their tools (of definition from, and interpretation into their roles) are defined as abbreviations of some fixed expressions. Then the only necessary conversions (by functors) between objects playing the role of the same meta-object, will link various useful representations of a new object by old ones.

Formalization of operations and currying

The n-ary operations acting as n-ary operators between n sets, are formalized by:

• n domain functors (of little practical use);

• an $(n+1)$-ary operator of evaluation (evaluator), with arguments an n-ary operation f and its arguments $x_1, \cdots, x_n$, written $f(x_1, \cdots, x_n)$;

• an operation definer, binding n variables to their respective ranges on a term. The binary operation defined by the term (binary predicate) t with arguments x in E and y in F, is written $(E \ni x, F \ni y \mapsto t(x, y))$, abbreviated when $E = F$ as $(E \ni x, y \mapsto t(x, y))$.

The notion of operation can be represented as a class of functions, in the following way called *currying*. As an operation definer (binding n variables) we take the succession of n uses of the function definer (one for each variable to bind); and similarly as an evaluator, n uses of the function evaluator:

$$f = (E \ni x, F \ni y \mapsto t(x, y)) \simeq (E \ni x \mapsto (F \ni y \mapsto t(x, y))) = g$$
$$f(x, y) = g(x)(y) = t(x, y)$$

The intermediate function $g(x) = (F \ni y \mapsto t(x, y))$ with argument y, sees x as free and y as bound. But this breaks the symmetry between arguments and loses the data of F when E is empty. A formalization without these flaws will be possible using tuples (2.1.).

The formalization of n-ary relations involves an $(n+1)$-ary predicate of evaluation, and a definer binding n variables on a formula. They may either be represented by operations by translating Booleans into objects, or defined in curried form (when $n > 1$) by 1 use of the set-builder and $n - 1$ uses of the function definer.

1.11. Set generation principle

Bounded quantifiers give sets their fundamental role as ranges of bound variables, unknown by the predicate $\in$ which only sees them as classes. Technically, the bounded quantifier $(\exists \; \in \; , \;)$ can define the predicate $\in$ by $x \in E \Leftrightarrow (\exists y \in E, x = y)$ but cannot be defined from it in return without open quantifier. Philosophically, the perception of a set as a class (classifying each object as belonging to it or not) does not provide its full perception as a set (the perspective over all its elements as coexisting).

Set generation principle. *For any class $\mathcal{C}$ defined by a bounded formula with parameters, if the expression $(\forall x, \mathcal{C}(x) \Rightarrow \mathcal{R}(x))$ of $\forall_\mathcal{C}$ on an undefined (extra symbol of) unary predicate $\mathcal{R}$ is proven equivalent to a bounded formula (here abbreviated like a quantifier Q as $Qx, \mathcal{R}(x)$), then $\mathcal{C}$ is a set that can be named by a new operator symbol K to be added to the language of set theory, with arguments the common parameters of $\mathcal{C}$ and Q, and axioms :*

(For all accepted values of parameters), $\mathrm{Set}(K) \wedge (\forall x \in K, \mathcal{C}(x)) \wedge (Qx, x \in K)$.

The equivalence between Q and $\forall_\mathcal{C}$ is expressible as follows, where $(\overline{Q}x, \mathcal{R}(x)) \not\Leftrightarrow (Qx, \neg\mathcal{R}(x))$:
(1) $\forall x, (\mathcal{C}(x) \Leftrightarrow \overline{Q}y, x = y)$ (in fact we just need $\forall x, \mathcal{C}(x) \Rightarrow \overline{Q}y, x = y$)
(2) $Qx, \mathcal{C}(x)$
(3) For all (undefined extra symbols of) unary predicates $\mathcal{A}$ and $\mathcal{B}$, if $\forall x, \mathcal{A}(x) \Rightarrow \mathcal{B}(x)$ then $(Qx, \mathcal{A}(x)) \Rightarrow (Qx, \mathcal{B}(x))$.

Indeed these 3 properties are already known consequences of "$Q = \forall_\mathcal{C}$". Conversely,
$((2) \wedge (3)) \Rightarrow ((\forall_\mathcal{C} x, \mathcal{R}(x)) \Rightarrow Qx, \mathcal{R}(x))$
$((1) \wedge \exists_\mathcal{C} x, \mathcal{R}(x)) \Rightarrow \exists x, ((\overline{Q}y, x = y) \wedge (\forall y, x = y \Rightarrow \mathcal{R}(y))) \Rightarrow ((3) \Rightarrow \overline{Q}x, \mathcal{R}(x))$. $\square$

(3) will often be immediate, by lack of any troubling occurrence of $\mathcal{R}$ in Q (negation, equivalence, left of a $\Rightarrow$), leaving to verify (1) and (2).

Here are examples of such operator symbols (denoting $D = \mathrm{Dom}\, f$):

K	$\{y \in E \mid \mathcal{B}(y)\}$	$\bigcup E$	$\mathrm{Im}\, f$	$\emptyset$	$\{y\}$	$\{y, z\}$
dK	$\forall x \in E, d\mathcal{B}(x)$	$\mathrm{Set}(E) \wedge \forall x \in E, \mathrm{Set}(x)$	$\mathrm{Fnc}(f)$			
$\mathcal{C}(x)$	$x \in E \wedge \mathcal{B}(x)$	$\exists y \in E, x \in y$	$\exists y \in D, f(y) = x$	0	$x = y$	$x = y \vee x = z$
$Qx, \mathcal{R}(x)$	$\forall x \in E, \mathcal{B}(x) \Rightarrow \mathcal{R}(x)$	$\forall y \in E, \forall x \in y, \mathcal{R}(x)$	$\forall x \in D, \mathcal{R}(f(x))$	1	$\mathcal{R}(y)$	$\mathcal{R}(y) \wedge \mathcal{R}(z)$
$\overline{Q}x, \mathcal{R}(x)$	$\exists x \in E, \mathcal{B}(x) \wedge \mathcal{R}(x)$	$\exists y \in E, \exists x \in y, \mathcal{R}(x)$	$\exists x \in D, \mathcal{R}(f(x))$	0	$\mathcal{R}(y)$	$\mathcal{R}(y) \vee \mathcal{R}(z)$

The definition of $K = \{x \in E \mid \mathcal{B}(x)\}$, that was only expressed as a class, can also be written as $((\forall x \in K, x \in E \wedge \mathcal{B}(x)) \wedge (\forall x \in E, \mathcal{B}(x) \Rightarrow x \in K))$, or as $(K \subset E \wedge \forall x \in E, x \in K \Leftrightarrow \mathcal{B}(x))$.

The functor $\bigcup$ is the union symbol, and its axioms form the *axiom of union*.

The set $\mathrm{Im}\, f$ of values of $f(x)$ when x ranges over $\mathrm{Dom}\, f$, is called the *image* of f.

We define the predicate $(f : E \to F) \Leftrightarrow (\mathrm{Fnc}(f) \wedge \mathrm{Dom}\, f = E \wedge \mathrm{Set}(F) \wedge \mathrm{Im}\, f \subset F)$, that reads "$f$ is a function from E to F". A set F such that $\mathrm{Im}\, f \subset F$ (i.e. $\forall x \in \mathrm{Dom}\, f, f(x) \in F$), is called a *range of* f. The more precise $(\mathrm{Fnc}(f) \wedge \mathrm{Dom}\, f = E \wedge \mathrm{Im}\, f = F)$ will be denoted $f : E \twoheadrightarrow F$ (f is a *surjection*, or *surjective function* from E to F, or function from E *onto* F).

The *empty set* $\emptyset$ is the only set without element, and is included in any set E ($\emptyset \subset E$).

Thus, $(E = \emptyset \Leftrightarrow E \subset \emptyset \Leftrightarrow \forall x \in E, 0)$, and thus $(E \neq \emptyset \Leftrightarrow \exists x \in E, 1)$.

This constant symbol $\emptyset$ ensures the existence of a set; for any set E we also get $\emptyset = \{x \in E \mid 0\}$.

We can reobtain $\exists$ by $(\exists x \in E, \mathcal{R}(x)) \Leftrightarrow \{x \in E \mid \mathcal{R}(x)\} \neq \emptyset \Leftrightarrow (1 \in \mathrm{Im}(E \ni x \mapsto \mathcal{R}(x)))$.

As $(\mathrm{Dom}\, f = \emptyset \Leftrightarrow \mathrm{Im}\, f = \emptyset)$ and $(\mathrm{Dom}\, f = \mathrm{Dom}\, g = \emptyset \Rightarrow f = g)$, the only function with domain $\emptyset$ is called the *empty function*.

For all x, $\{x, x\} = \{x\}$. Such a set with only one element is called a *singleton*.

For any x, y we have $\{x, y\} = \{y, x\}$. If $x \neq y$, the set $\{x, y\}$ with 2 elements x and y is a *pair*.

Our set theory will later be completed with more symbols and axioms, either necessary (as here) or optional (opening a diversity of possible set theories).

Philosophical aspects of the foundations of mathematics (The flow of the meta-mathematical time)

Let us complete our initiation to the foundations of mathematics, by some notes on their philosophical and intuitive aspects usually not so well explained though somehow implictly understood by specialists (as philosophical issues are not easily seen as proper objects of scientific works).

In particular, we will explain

- How, while independent of our time, the universe of mathematics is still subject to its own time ;
- The deep meaning of the difference between sets and classes, in relation to that time ;
- Thus, the meaning behind the difference of syntax of quantifiers between set theory and generic (first-order) theories; and so the difference between open and bounded quantifiers
- The justification of the set generation principle

These philosophical complements are not really needed for the next section (2. Set Theory, continued), except to explain the meaning behind the fact that the powerset axiom (2.6) cannot be deduced from the set generation principle ; this meaning and its consequences on the special status of the powerset, will be further investigated in text 3 (Model theory).

Intuitive representation and abstraction

Though mathematical systems "exist" independently of any particular sensation, we need to represent them in some way (in words, formulas or drawings). Diverse ways can be used, that may be equivalent (giving the same results) but with diverse degrees of relevance (efficiency) that may depend on purposes. Ideas usually first appear as more or less visual intuitions, then are expressed as formulas and literal sentences for careful checking, processing and communication. To be freed from the limits of a specific form of representation, the way is to develop other forms of representation, and exercise to translate concepts between them. The mathematical adventure itself is full of plays of conversions between forms of representation.

Platonism vs Formalism

In this diversity of approaches to mathematics (or each theory), two trends can be philosophically distinguished: a Platonic and a formalistic view.

The *Platonic* view (also called *idealistic*) focuses on the worlds or systems to study, seen as preexisting mathematical realities to be explored (Plato called that a remembering). This is the approach of intuition that smells the global order of things before formalizing them.

The *formalistic* view focuses on language, rigor and dynamical aspects of a theory, starting from its foundation (its formal expression), and following the rules of development.

Philosophers usually present them as opposite, incompatible belief systems, candidate truths on the real nature of mathematics. However, they instead turn out to be both necessary and complementary aspects of its foundations. Let us explain how.

Of course, human thought having no infinite abilities, cannot fully operate in a realistic way, but its works can (at least roughly) be expressed as a formal development from some foundation (which prevents the errors of intuition). But the formalistic view cannot absolutely hold either because

- The clarity and independence of any possible foundation (any starting position with formal development rules from it), remain relative: any starting point had to be intuitively chosen somehow arbitrarily, and/or motivated by a larger perspective, among mathematical realities; it must be described in some presumably meaningful way, implicitly admitting its own foundation, since any try to specify the latter would lead to a path of endless regression, whose realistic preexistence would have to be admitted.
- Most of the time, practical works are only partially formalized: we use semi-formal proofs, with just enough rigor to give the feeling that a full formalization is possible, yet not fully written; an intuitive vision of a problem may seem clearer than a formal argument.

Another reason for their reconciliation, is that they are not in any global dispute to describe the whole of mathematics, but their shares of relevance depends on the specific theories being considered.

Realistic vs. axiomatic theories in mathematics and other sciences

Interpretations of the word "theory" may vary between mathematical and non-mathematical uses (in ordinary language and other sciences), in two ways:

Theories may differ by their object and nature:

- **Pure mathematical theories**, are mathematical theories considered for the pure sake of mathematics, without any non-mathematical intentions.

On the contrary, theories outside mathematics try to describe some real systems (fields of observation, parts of the outside world, that are not purely mathematical systems). They may be of 2 sorts:

- **Applied mathematical theories** are also mathematical theories (i.e. rigorously expressed) but the mathematical systems they describe are conceived as idealizations of the given real systems (focusing on some hopefully definite mathematical structure, neglecting other aspects); if successfully exact, this idealization also allows for exact deductions without risk to depart from accepted margins of error.
- **Non-mathematical theories** describing qualitative (non-mathematical) aspects. For example, usual descriptions of chemistry involve drastic approximations, recollecting from observations some seemingly arbitrary effects whose deduction from quantum physics are usually out of reach of calculations.

Theories may also differ by whether Platonism or formalism best describes their intended meaning:

A **realistic theory** aims to describe a **fixed system** given from an independent reality, so that any of its formulas (claims) will be either definitely true or false as determined by this system (but the truth of a claim outside mathematics may be ambiguous, i.e. ill-defined for the given real system). Over this intention, the theory will be built by providing an initial list of formulas called axioms : that is a hopefully true description of the intended system as currently known or guessed. Thus, the theory will be true if all its axioms are indeed true on the intended system.

This is usually well ensured in pure maths, but may be speculative in other fields. In realistic theories outside pure mathematics, the intended reality is usually contingent among alternative possibilities, that (for applied mathematical theories) are equally possible from a purely mathematical viewpoint. To find out which theory (i.e. axioms list) fits the specific reality that pure mathematics cannot identify, scientific theories need to be falsifiable (false claims should run the risk of being refuted by observations). But in pure mathematics, the usual details of both possible intended roles of theories (realistic and axiomatic) automatically protect them from the risk to be meaningfully "false" as long as the formal rules are respected.

An **axiomatic theory** is a theory given with an axioms list that means to **define** the selection of the range of considered models. This selection may keep an unlimited diversity of possible models, that remain equally real and legitimate interpretations. By this definition of what model means, the truth of the axioms of the theory is automatic (it holds by definition and is thus not questionable) on each of its models. All theorems (logical consequences of the axioms) are also true in each model. (Thus in particular, when a realistic theory is true, its theorems will also be true on the intended model.)

Non-realistic theories outside pure mathematics (where the requirement of truth of theorems is not always strict) may either describe classes of real systems, or be works of fiction describing imaginary or possible future systems. But this distinction between real and imaginary systems does not exist in pure mathematics, where all possible systems are equally real. Thus, axiomatic theories of pure mathematics aim to describe a mathematical reality that is existing (if the theory is consistent) but generally not unique.

Different models may be non-equivalent, in the sense that some formulas may be true or false depending on the model. Formally, these formulas are undecidable, i.e. they cannot either be proven nor refuted from the axioms (a refutation means a proof of the negation). Different consistent theories may "disagree" without conflict, by being all true descriptions of different systems, that may equally "exist" in a mathematical sense without any issue of "where they are".

For example Euclidean geometry, in its role of first physical theory, is but one in a landscape of diverse geometries that are equally legitimate for mathematics, and the real physical space is better described in thinner details by the non-Euclidean geometries given by Special and General relativity. Similarly, biology is relative to a huge number of random choices secretly accumulated by Nature on Earth during billions of years.

Realistic and axiomatic theories both appear in pure mathematics, in different parts of the foundations of mathematics. A first sketch of explanation of this fact is presented in the section on truth in mathematics, after exploring the roles of a purely mathematical flow of time (independent of our time) in model theory and set theory.

Time in model theory

Time of interpretation [see 1.5]

For any model M of a theory T, the model $[T, M]$ of one-model theory actually has 3 parts:

- The components of the theory: the symbolic system (types, symbols, formulas, proofs…), that aims to describe the model but remains outside it and independent of it. As such it has no meaning.
- The model itself: the system of objects and structures connecting them, described by the theory.
- The interpretation of expressions in the model, for any values of their free variables.

The process of interpretation of all expressions of a given theory in a given system, is a mathematical construction determined by the combination of both systems of data (the theory and its studied system) but it is not directly contained in these data. Instead, it forms another system, larger than the first one, and built *after* it.

The metaphor of the usual time

The world of objects being studied is the past (in the universe of mathematics). Our act of interpreting expressions there, forms the present.

Describing the world and the way to describe it, means describing everything, and something beyond it. Mentioning "what I mean", does not itself say what it is, as it might be anything, and becomes absurd if the phrase modifies or contradicts it ("the opposite of what I'm saying").

Mentioning “what I will tell about tomorrow”, even if I knew the words I will pronounce, would not suffice to already provide their meaning either: ifever I will mention “what I told about yesterday” (thus now) it would make a vicious circle; but even if the structure of my future words would ensure that its meaning will exist tomorrow, it would still not provide it today. While I may speculate on it, its actual meaning will only arise once actually expressed in context.

The lack of interest to discuss words without their meaning, makes it better to ignore the present and future sayings, and focus on the description of past ones. Like historians, I can only describe the universe of the past, being myself in a present outside this universe. I can speak of “what I told about at that time”: it has a sense if those past words already had one, because I got that meaning and I remember it.

My current universe of the past that I can describe today, includes the one of yesterday, but also my yesterday’s comments about it and their meaning. So I can describe today things outside the universe I could describe yesterday. Meanwhile I neither learned to speak Martian nor acquired a new transcendental intelligence, but the same language applies to a broader universe with new objects. As these new objects are of the same kind as the old ones, my universe of today can resemble my universe of yesterday; but from one universe to another, the same words can take a different meaning.

The finite time between expressions

With a fixed model, expressions do not all at once receive their interpretation, but only the ones after the others because these interpretations depend on each other, thus must be calculated after each other. This time order of interpretations between expressions, follows the hierarchical order between expressions and their subexpressions.

Take for example, the formula $xy + x = 3$. In order for it to make sense, the variables x and y must take a value first. Then, xy takes a value, obtained by multiplying the values of x and y. Then, $xy + x$ takes a value out of the previous ones. Then, the whole formula ($xy + x = 3$) takes a boolean value (true or false). But this value depends on those of the free variables x and y. Finally, taking for example the closed formula $\forall x, \exists y, xy + x = 3$, its Boolean value (which is false in the world of real numbers), “is calculated from” those taken by the previous formula for all possible values of its variables, and therefore comes *after* them.

The infinite time between theories

A finite list of formulas in a theory may be interpreted by a single big formula containing them all. This big formula thus comes (or : is interpreted) after them all, but still belongs to the same theory. But for only one formula to interpret an infinity of formulas (possibly all of them, handled as values of a variable), this requires to switch to the framework of the one-model theory whose model encompasses the previous one, with all its formulas and their interpretation. So it belongs to another theory, that comes after the first one.

The larger world containing all interpretations of formulas of the current theory in the current model, is the next universe of the past, as it will be once the current infinite sequence of interpretations of all expressible formulas seen as describing the present world of past objects, will become past.

Or can it be otherwise ? Would it be possible for the current theory to express or at least simulate the notion of its own formulas and compute their values ?

At least some theories include objects that represent formulas. In particular we can have a theory T where each model contains an infinite set of objects among which all possible formulas of T itself are figured. However, the Truth Undefinability theorem will show that even if T similar to the object-theory, there is no possible way (any definition by a formula) to attribute Boolean values to all object-formulas, that will agree with all values taken by formulas of T in this model.

This infinite time between theories, will turn out to develop as an endless hierarchy of infinities.

Time in set theory

Zeno’s Paradox as a metaphor of the metamathematical time

Achilles runs after a turtle; whenever he crosses the distance to it, the turtle takes a new length ahead.
Seen from a height, a vehicle gone on a horizontal road approaches the horizon.
Particles are sent in accelerators closer and closer to the speed of light.
Can they reach their ends ?

Each example can be read in two ways: the “closed” view, sees a reachable end; the “open” view ignores this end, but only sees the movement towards it, never reaching it. Finally, the “true” interpretation in each case is given by some “physical measure” of the cost to reach the end, telling whether this cost would be finite or infinite. But the world of mathematics, that knows no cost and where objects only play arbitrarily defined roles, can accept both interpretations.

Each generic theory is "closed", as it can see its world (the range of its variables) as a set, and can prove any consequence of its axioms. But any possible foundation for it (either by one-model theory or by set theory) escapes this whole. Hence the need for an open theory integrating each world (model) described by a theory, as a part of a later world, forming an endless sequence of growing realities. But this is already the role of set theory with its way to only bind variables on sets [1.8].

The relative sense of open quantifiers

As an exclusion of ends, set theory excludes open quantifiers from its formulas: once the free variables and the contents of involved sets are fixed, its bounded expressions (expressing local, explicitly subjective questions) make sense regardless of rest of the universe. So, while the interpretation of a generic theory requires to fix a model for giving values to its variables and expressions, set theory is meant to possibly fix the values of its free variables and locally interpret its expressions independently of the rest of the universe that may keep growing (only constrained by the chosen axioms).

By not always giving sense to its formulas with open quantifiers, set theory acknowledges the subjectivity of their interpretation, that may only be given "here and now" : a universal claim $(\forall x, \cdots)$ true "here", may become false (find a counterexample) "elsewhere". But while the existence of different places alternatively giving relative truth and falsity to a formula implies its indefiniteness, the converse may not hold. Indeed, any claim about the range of possible views (how can actually things be "elsewhere"), would be relative to, and thus would require to specify, the variation range of the universe. But a general claim on this range, such as "$(\forall x, \cdots)$ is true in all universes", would amount to describe their union : "$(\forall x, \cdots)$ is true in the union of all universes", which would just be another specific (supposedly ultimate) universe. But no careful choice of an axiomatic theory trying to describe the ultimate universe (as if such a concept made any sense), can ever decide (prove or refute) all formulas.

The Completeness theorem links this indeterminacy to the limits of formal provability: the formal undecidability of some $(\exists x, \mathcal{A}(x))$ means it is false in some universe $\mathcal{U}$ but true in another universe $\mathcal{U}'$. In the case where $\mathcal{U} \subset \mathcal{U}'$, this means that $\mathcal{A}(x)$ is only true for some $x \in \mathcal{U}'$ that are all outside $\mathcal{U}$. We can intuitively say that these values of x cannot be formally expressed, and their existence cannot be deduced from any accepted existence axiom.

However, while the formal decidability (provability or refutability) of a formula suffices to make it definite, the converse does not always morally hold either. Namely, even if some $(\exists x, \mathcal{A}(x))$ was formally irrefutable (and even if this fact itself was provable), this would still not always morally imply the ultimate truth of that existence formula (as referring to a possible extension of the current universe), because there are other possibilities: a given universe where $(\exists x, \mathcal{A}(x))$ is false may not be part of any another one where it is true. Indeed, while such another universe may exist, both may not be able to coexist as parts of a common larger universe without losing some properties, as we shall see (with non-standard models and the incompleteness theorem, in section 3) that they may disagree on arithmetic and powerset.

Similarly, if $(\exists x, \mathcal{A}(x))$ is unprovable, a given universe where it is true might not contain any of those where it is false among its classes. We can intuitively say that different possible universes do not always follow one another in time, but may belong to separate, incompatible growth paths ; and existing models of consistent theories are not all "faithful pictures of reality". Thus, the indeterminacy of open formulas should only be treated by avoidance, understood as a mere expression of ignorance towards the range of acceptable universes, partially selected by axioms, among which these formulas may take diverse values.

Nature of classes

The equality between classes $\mathcal{A}$ and $\mathcal{B}$ would be defined by $\forall x, \mathcal{A}(x) \Leftrightarrow \mathcal{B}(x)$. In particular, any universal claim $(\forall x, \mathcal{A}(x))$ means the equality of $\mathcal{A}$ to the universe. For the intended interpretation of set theory (with a growing universe), the meta-relation of equality between classes is as indefinite as the open $\forall$, thus must be abandoned : classes do not behave like objects. But we still have a notion of "proven equality" between $\mathcal{A}$ and $\mathcal{B}$, as given by proofs of the formula $(\forall x, \mathcal{A}(x) \Leftrightarrow \mathcal{B}(x))$.

Each universe $\mathcal{U}$ sees each class $\mathcal{C}$ as a meta-set of objects $P = \{x \in \mathcal{U} | \mathcal{C}(x)\}$; and calls it a set if it has the same elements as an object "set", $P \in \mathcal{U}$. Formally, $(\exists P, \forall x, \mathcal{C}(x) \Leftrightarrow x \in P)$, or equivalently $(\exists E, \forall x, \mathcal{C}(x) \Rightarrow x \in E)$ as this can provide $P = \{x \in E | \mathcal{C}(x)\}$. Otherwise, this P is created now and will exist as a set in future universes. But this will still not make $\mathcal{C}$ a set if its interpretations as meta-subsets of future universes will differ from its current value as meta-subset P of our $\mathcal{U}$.

From the perspective of a variable universe, a class $\mathcal{C}$ "is a set" (equal to P) if it remains constant as a meta-set (P that depends on $\mathcal{U}$) during the growth of $\mathcal{U}$. Precisely, it is known as a set (proven equal to P) if we could prove this independence, i.e. exclude the possibility for any object x outside our current universe $\mathcal{U}$ (but existing in some extension of $\mathcal{U}$), to ever satisfy $\mathcal{C}(x)$. On the contrary, a class $\mathcal{C}$ that is not a set remains able to eventually contain "unknown" or "not yet existing" objects, that will be added to P (will belong to its future versions) and make it grow during the growth of $\mathcal{U}$.

If we choose to give the growth of $\mathcal{U}$ some "fixed range", then the formula $(\exists P, \forall x, \mathcal{C}(x) \Rightarrow x \in E)$ that qualified $\mathcal{C}$ as a set in each universe, once interpreted in the "ultimate" universe (union of these $\mathcal{U}$), turns out to express that in this growth, "there is a time after which P is constant" (thus ignoring any past variations, to focus on the latest ones). But as the variable $\mathcal{U}$ escapes all sets, every perspective can be surpassed by another, and a class defined by some special formula might alternatively gain and lose the status of set (but if in some growth range, "P perpetually alternates between variability and constancy", then it would ultimately not be constant there, thus $\mathcal{C}$ would not be a set).

Concrete examples

A set: Is there any dodo left on Mauritius ? As this island is well known and regularly visited since their supposed disappearance, no surviving dodos could still have gone unnoticed, wherever they may hide. Having not found any, we can conclude there are none. This question, expressed by a bounded quantifier, has an effective sense and an observable answer.

A set resembling a class: Bertrand Russell raised this argument criticizing theology : "If I were to suggest that between the Earth and Mars there is a china teapot revolving about the sun..., nobody would be able to disprove my assertion [as] the teapot is too small to be revealed even by our most powerful telescopes. But if I were to go on to say that, since my assertion cannot be disproved, it is intolerable presumption on the part of human reason to doubt it, I should rightly be thought to be talking nonsense." The question is clear, but on a too large set, making the answer inaccessible.

A class: the extended claim, "there is a teapot orbiting some star in the universe" loses all meaning: not only the size of the universe is unknown, but Relativity theory considers the remote events from which we did not receive light yet, as not having really happened yet either.

A meta-object: how could God "exist", if He is a meta-object, while "existence" can only qualify objects? Did apologists properly conceive their own thesis on God's "existence" ? But what are the objects of their faith and their worship ? Each monotheism rightly accuses each other of only worshiping objects (sin of idolatry): books, stories, beliefs, ideas, feelings, places, events, miracles, sufferings, errors, diseases, disasters (declared God's Will), hardly more subtle than old statues, not seriously checking (by fear of God) any hints of their divinity.

A universal event: the redemptive sacrifice of the Son of God. Whether it would have been theologically equivalent for it to have taken place not on Earth but in another galaxy or in God's plans for the Earth in year 3,456, remains unclear.

Another set reduced to a class... The class F of girls remains incompletely represented by sets: the set of those present at that place and day, those using this dating site and whose parameters meet such and such criteria, etc. Consider the predicates B of beauty in my taste, and C of suitability of a relationship with me. When I try to explain that "I can hardly find any pretty girl as I need (and they are often unavailable anyway)", i.e.

$$(\forall_F x, C(x) \Rightarrow B(x)) \wedge \{x \in F | B(x)\} \approx \emptyset,$$

the common reaction is: "Do you think that beauty is the only thing that matters ?", i.e.

$$\text{What,}(\forall_F x, C(x) \Leftrightarrow B(x))????$$

then "If you find a pretty girl but stupid or with bad character, what will you do ?". Formally : $(\exists_F x, B(x) \not\Rightarrow C(x)$!!!). And to conclude with a claim of pure goodness: "I am sure you will find !", ie : "$\exists\exists x \in F, C(x)$", but under one necessary condition : "You must change your way of thinking".
... **by the absence of God**... : F would have immediately become a set by the existence of anybody on Earth able to receive a message just composed by God's will, as He should obviously have used this chance to make him email me the address my future wife (or mine to her).
... **and of any substitute**: a free, open and efficient online dating system, as would be included in my project trust-forum.net, could give the same result. But this requires finding programmers willing to implement it. But the class of programmers is not a set either, especially as the moral purpose of the project does not fit with the religious moral priority of preserving God from any risk of unemployment, to secure His salary of praise.

Can a set contain itself ?

Let us call a set *reflexive* if it contains itself. Russell's paradox [1.9], used the class of non-reflexive sets to show that it cannot be a set. But the existence of reflexive sets is undecidable, here is why.

We can try to see it unprovable, by deleting all reflexive sets: they would not be reconstructible from the data of their elements, but we would still need to delete any other set they would belong to, and so on. Rather, we can progressively rebuild the universe while avoiding them: each set is formed at some time, as a gathering of pre-existing (previously formed) elements and sets (from some previous "universe"), to constitute the next universe. But as a set once produced in such a way did not yet exist before its creation, it was not among the elements that formed it, thus it cannot belong to itself either. This is independent of context: a union of universes each devoid of reflexive element, cannot contain one either.

On the other hand, universes with reflexive sets can be created as follows:

Riddle: What is the difference between

- A universe with a pure element x and a set y such that $x \in y$ and $y \notin y$
- A universe with a set x and a pure element y such that $x \in x$ and $y \notin x$?

Answer: the role of the set containing x but not y, played by y in the former universe, is played by x in the latter.

The absence of reflexive sets, is a special case of the axiom of foundation, that is also undecidable for the same reasons. This axiom and its undecidability will be explained in terms of well-founded relations (after the study of Galois connections). But it is just as useless as the sets it excludes.

Justifying the set generation principle

Let $\overline{Q}$ be an equivalent expression of $\exists_{\mathcal{C}}$. Namely, let $\overline{Q}y, \mathcal{R}(y)$ be the abbreviation of a quantifying formula (a formula that can use an extra symbol $\mathcal{R}$ of unary predicate) such that $\neg(\overline{Q}y, 0)$, and let $\mathcal{C}(x)$ be proven equivalent to the formula $\overline{Q}y, y = x$. This formula $\overline{Q}y, \mathcal{R}(y)$ has only definite, fixed means (variables bound to given sets, fixed parameters) to provide values to the argument y on which its predicate $\mathcal{R}$ is ever tested in its sub-formulas. These values thus range over a set E. Then $\mathcal{C}$ is a set because for all x, $\mathcal{C}(x) \Rightarrow ((\overline{Q}y, y = x) \not\Leftrightarrow (\overline{Q}y, 0)) \Rightarrow (\exists y \in E, y = x \not\Leftrightarrow 0) \Rightarrow x \in E$. □

For classes satisfying the condition of the set generation principle (being indirectly as usable as sets in the role of domains of quantifiers), are they also indirectly as usable as sets in the role of domains of functions (before using this principle) ? Namely, is there for each such class a fixed formalization (bounded formulas with limited complexity) playing the roles of definers and evaluators for functions having these classes as domains ? The answer is yes, but we shall not detail the justifications here.

Concepts of truth in mathematics

I see 4 concepts of "truth" in mathematics, here presented from the simplest to the most subtle.

We first saw the **relative truth**, that is the interpretation of a formula in a model, as defined in model theory. In this sense a given formula may be as well true or false depending on where it is interpreted (but the "choice" of model can remain an abstract variable, not necessarily something we can construct), and on the values of explicit free variables of the formula (if it has any).

Then comes **provability in first-order logic**, which is crucial in the foundations of mathematics, since all mathematics is expressible in this framework (via the formalization of set theory as a first-order theory). A reconciliation between Platonism and formalism will be obtained by Gödel's Completeness theorem (3.1), establishing the real existence of models of any consistent first-order theory. It will show that formal undecidability precisely fits the diversity of possible truths depending on models (provability is equivalent with "relative truth in all models").

Thus, the (equivalent) notions of "proof", "theorem" and "consistency" themselves turn out to have a clear realistic sense, as there is an essentially unique and universal solution of what the word "proof" really means as a formal system (expressible as a proof verification algorithm). Here, "essentially unique" means that all such formal systems (proof syntactic systems with their verification algorithms) are equivalent, i.e. any formal proof for the one is automatically convertible into a proof for the other. These notions of proof are meant in an ideal sense ignoring any physical limit of computing power, while it has been shown (see Gödel's speed-up theorem) that the shortest proofs of even relatively simple theorems may be bigger that our whole physical Universe.

The Completeness theorem will be proven by constructing models of consistent first-order theories by means of some infinite sets of all possible expressions that can be written in the theory. The existence of these sets, which this form of mathematical realism depends on, finally results from the mere actual infinity of the set of natural numbers, and is thus independent of the particular theory.

Then comes the **realistic truth of first-order formulas of arithmetic** (the theory of natural numbers), meant to be interpreted in its standard model $\mathbb{N}$, that is the "true set of all finite natural numbers". This is involved for making a realistic sense of the provability predicate, including for formulas for which "we did not find a proof yet" : the abstract claim of provability, that is the "existence of a proof", can be formalized as a first-order formula of arithmetic with one unbounded quantifier ($\exists p \in \mathbb{N}$, where p codifies the proof), and many bounded quantifiers ($\exists x < \cdots, \forall y < \cdots$).

The incompleteness theorem shows that this realistic theory cannot be completely formalized as an axiomatic theory in any algorithmic manner ; and this formal undecidability even affects the very formula or provability predicate (which was involved in the completeness theorem). But even though the axiomatization of arithmetic cannot be completed, it can be endlessly extended beyond any given candidate.

The natural method to progress towards formalizations of arithmetic that can correctly decide more arithmetical formulas, consists in developing stronger axiomatizations of set theory, including descriptions of higher and higher infinities (cardinals) above the infinity of $\mathbb{N}$, and inside which the set $\mathbb{N}$ is studied as a particular case.

Actually, the properly understood intention of set theory is neither axiomatic nor realistic, but a fuzzy intermediate between both. Indeed, it aims to picture "the mathematical universe" as being as big as possible, but no fixed mathematical universe can ever be the ultimate one. Instead, the mathematical universe has the ability of perpetually expanding, during its own time (unrelated to our ordinary human or physical time), beyond any (temporarily) fixed totality that could once be thought of. The descriptions of successive universes (models of set theory) by first-order formulas during this expansion, can never reach any stable limit (as an axioms system that could be taken as the description of "the ultimate universe"). Formally, no fixed conception that can be made of the mathematical reality (an algorithmically defined axiomatic set theory), can ever exhaust the possible information about this reality, even among mere first-order arithmetical formulas.

Thus, the last useful concept of truth, is the **truth of set theoretical axioms**, that aim to make the set theoretical universe as big as possible and thus expand (but never exhaust) the range of formal decidability of first-order arithmetical formulas. Such an axiomatic system needs to be chosen carefully, with hopefully good philosophical reasons to be considered legitimate, in such a way that its contributions to the provability of arithmetical formulas are expected to keep conformity with the real arithmetical truths (in particular the theory remains consistent), and at the same time we wish it to be powerful (proving as many arithmetical truths as possible).

Fortunately, relatively simple but very powerful theories (such as ZF) suffice to describe quite sufficiently large realities for most needs, so as to work in practice as if the universe of all mathematical objects was a fixed, absolute and eternal reality, even though it cannot be so in details. A Platonic view of set theory can be a good approximation, but not an absolute fact.

Alternative logical frameworks

The description made here of the foundations of mathematics (first-order logic and set theory), is essentially just an equivalent clarified expression of the widely accepted ones (a different introduction to the same mathematics). In section 3 will be presented other logical frameworks that are either restricted versions of first-order logic, or anyway naturally convertible into the framework of set theory as we did. But other, more widely different frameworks (concepts of logic and/or sets), called *non-classical logic*, might be considered.

• Some logicians developed the "intuitionistic logic", which lets formulas keep a possible indefiniteness as we mentioned for open quantifiers, but treated as a modification of the pure Boolean logic (the rejection of the excluded middle, where $\neg(\neg A)$ does not imply A), without any special mention of quantifiers as the source of this uncertainty. Or it might be seen as a formal confusion between truth and provability. There, $\{0\} \cup\ ]0,1] \subset [0,1]$, without equality. I could not personally find any interest in this formalism but only heard that theoretical computer scientists found it useful.

• When studying measure theory (which mathematically defines probabilities in infinite sets), I was inspired to interpret its results as simpler statements on another concept of set, with the following intuitive property. Let x be a variable **randomly** taken in $[0,1]$, by successively flipping a coin for each of its (infinity of) binary digits. Let E be the domain of x, set of all **random** numbers in $[0,1]$. It is nonempty because such random numbers can be produced. Now another similarly random number, with the same range ($y \in E$) but produced independently of x, cannot (has no chance to) be equal to x anymore: $\forall x \in E, \forall y \in E, x \neq y$. Thus $x \in E$ is no more always equivalent to $\exists y \in E, x = y$.

We will keep classical logic in all following sections, ignoring such alternatives.